Kolmogorov, forward,backward, fixation probability and time.

A discrete stochastic system is one where an observation(measure) of the system results in an integer (or a vector of integers). Some examples are number of neutrons in a nuclear reaction, number of microbes moving around and duplicating, number of molecules in a chemical reaction. Many concepts in stochastic systems are very simple when formulated for discrete processes and give rise to systems of ordinary differential equations (ODE). Very often, when these concepts are stated, the continuous limit is taken for large systems and the system of ODE transforms then into a single partial differential equations (PDE). The continuous limit may be problematic and should be derived carefully: often, the boundary conditions are naturally stated for discrete processes but have to be adjusted for continuous systems.

I. TRANSITION RATES.

Consider a one step stochastic process (extension to multi- step is trivial) with time independent transition probabilities

$$W(n \to n \pm 1) = W^{\pm}(n)$$

This is the probability density at which the system jumps from sate n to state $n \pm 1$ during a short (infinitesimal) time interval dt. Experimentally, it can be measured by observing a large number of time jumps from state n to state $n \pm 1$. The jump times are exponentially distributed with parameter

$$\tau_n = \left(W^+(n) + W^-(n)\right)^{-1} \tag{1}$$

Once a jump has occurred, the *probability* to be a $n \to n \pm 1$ event is

$$P_{\pm}(n) = P(n \to n \pm 1) = W^{\pm}(n) \left(W^+(n) + W^-(n)\right)^{-1} \tag{2}$$

Gillespie Algorithm.

This is indeed how the stochastic systems are simulated. The computer simulates one trajectory at a time. Beginning the system in state n at time $t = 0$, the system draw an exponential random variable to determine the time to the next transition:

$$t = t + \text{Poisson}(\tau_n)$$

and a uniform random variable $x \in [0, 1]$ to determine the direction of the transition

$$\begin{aligned} x &= \text{Uniform}(0,1); \\ \text{if}(x < P_-(n)) \ &\text{then} \ n = n - 1; \\ &\text{else} \ n = n + 1; \end{aligned}$$

The computer then loops the above algorithm to generate a trajectory, and then loop some more to generate *many* trajectories for a given time interval or until absorption has occurred (on this, more later).

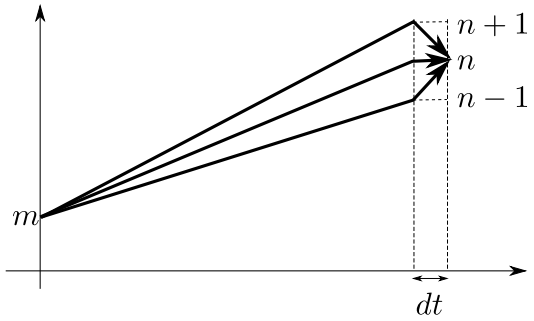

Figure 1: Forward equation.

II. FORWARD AND BACKWARD MASTER EQUATIONS.

Beginning with different states, the system will follow different stochastic paths and reach different states at the time t. Let us call $P(n, m; t)$ the probability density to reach state n beginning with state m at time t. This can be measured by observing (or simulating) a large number of trajectories during a time interval t, and selecting the ones which begin in m and ends in n ; $P(n, m; t)$ will be the relative number of these paths. We can relate this probability to adjacent probabilities, either at the final point or at the initial one. These will give rise to two *different* equations called forward and backward Kolmogorov equations.

A. Forward.

We will fix the initial state m. Let us decompose $P(n, m; t)$ into "and"s and "or"s. The probability to be in n at time t is (fig. 1)

- (the probability to be at $n-1$ at time $t-dt$) AND (jumping to n during dt) OR
- (the probability to be at $n+1$ at time $t-dt$) AND (jumping to n during dt) OR
- (the probability to be at n at time $t - dt$) AND (staying in n during dt):

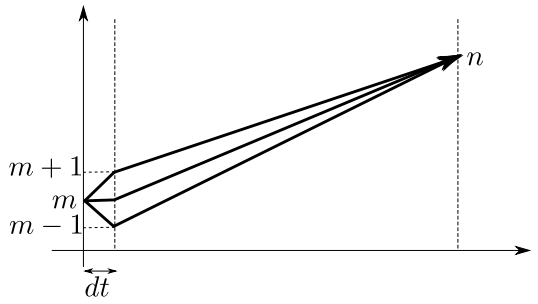

Figure 2: Backward equation.

$$\begin{aligned} P(n,m;t) &= P(n-1,m;t-dt)W^+(n-1)dt \\ &+ P(n+1,m;t-dt)W^-(n+1)dt \\ &+ P(n,m;t-dt)W^0(n) \end{aligned} \quad (3)$$

where $W^0(n)$ is the probability of staying in state n during the short interval dt:

$$W^0(n) = 1 - \left(W^+(n) + W^-(n)\right) dt$$

We can now develop equation (3) into power series of dt . Note that the first two lines don't have terms in $(dt)^0$. The 0-order term of the third lines cancels the left hand side of the equation and therefore

$$\begin{aligned} \frac{dP(n,m;t)}{dt} &= W^+(n-1)P(n-1,m;t) - W^+(n)P(n,m;t) \\ &+ W^-(n+1)P(n+1,m;t) - W^-(n)P(n,m;t) \end{aligned} \quad (4)$$

The interpretation of this Master equation is simple : the change in state n is the balance of upward and downward flux into it.

B. Backward.

In a manner analog to the forward equation, we can relate $P(n,m;t)$ to adjacent probabilities at the initial point(fig. 2):

$$\begin{aligned} P(n,m;t) &= P(n,m+1;t-dt)W^+(m)dt \\ &+ P(n,m-1;t-dt)W^-(m)dt \\ &+ P(n,m;t-dt)W^0(m) \end{aligned} \quad (5)$$

Note that, in contrast to eq.(3) each line contains the factor $W^i(m)$: this is because we *know* the initial state, contrary to the previous case where the state at time $t-dt$ was not known. Developing again in power of dt up to the first order, we get

$$\begin{aligned} \frac{dP(n,m;t)}{dt} &= W^+(m)\left[P(n,m+1;t) - P(n,m;t)\right] \\ &+ W^-(m)\left[P(n,m-1;t) - P(n,m;t)\right] \end{aligned} \quad (6)$$

Note, in contrast to the forward equation, how the transition rates appear in the backward equation.

C. Bounded systems.

From know on, we consider only bounded stochastic systems, *i.e.* the system will remain always between two boundary states which we will call 0 and N ($N > 0$). This statement, in terms of transition rates means that

$$W^+(N) = W^-(0) = 0 \quad (7)$$

These conditions modify slightly the boundary conditions for forward and backward Kolmogorov equation. For the forward equation and the flux into state $n = N$ ($n = 0$) , there is no upper (lower) path (fig.1) and we have

$$\frac{dP(N,m;t)}{dt} = W^+(N-1)P(N-1,m;t) - W^-(N)P(N,m;t) \quad (8)$$

$$\frac{dP(0,m;t)}{dt} = W^-(1)P(1,m;t) - W^+(0)P(0,m;t) \quad (9)$$

Of course because of the condition (7), we don't have to write these boundary conditions explicitly, but it is more clear to state them anyway. The same considerations apply to the backward equation:

$$\frac{dP(n,N;t)}{dt} = W^-(N)\left[P(n,N-1;t) - P(n,N;t)\right] \quad (10)$$

$$\frac{dP(n,0;t)}{dt} = W^+(0)\left[P(n,1;t) - P(n,0;t)\right] \quad (11)$$

III. FIXATION PROBABILITIES.

Suppose now that both boundary states $n = N$ and $n = 0$ are absorbing : once the system has reached one of these states, it will remain there. If we think for example, in terms of two competing microbial species, once one species has gone extinct, it will never come back. Making these states absorbing means that

$$W^-(N) = W^+(0) = 0$$

Such a system will eventually absorb into either $n = 0$ or $n = N$ state:

$$\lim_{t\to\infty} P(0,m;t) + P(N,m;t) = 1$$

An important question is: beginning in state m, what it the probability

$$\pi_N(m) = P(N,m;\infty)$$

of being eventually adsorb into state N ? In other terms, which proportion of trajectories will reach $n = N$ instead of $n = 0$.

Obviously, if we know this quantity, we also know $\pi_0(m) = 1 - \pi_N(m)$. To answer this question, we again decompose the probability with "and"s and "or"s and then, will use the backward Kolmogorov to find a

useful relation. Let us first note that $\pi_N(0) = 0$ and $\pi_N(N) = 1$. We therefore look for other cases $m \neq 0, N$. In order to get to state N, we have to get out of state m (and into $m \pm 1$) and then to N. The probability that the transition is into $m \pm 1$ is given by eq(2) and therefore

$$\begin{aligned} \pi_N(m) &= \frac{W^-(m)}{W^-(m)+W^+(m)}\pi_N(m-1) \\ &+ \frac{W^+(m)}{W^-(m)+W^+(m)}\pi_N(m+1) \end{aligned}$$

calling $w^{\pm}(m)$ the (normalized) rates in the above equation, we get a tri-diagonal linear system:

$$-w_m^-\pi_{m-1} + \pi_m - w_m^+\pi_{m+1} = 0 \quad (12)$$

$$m = 1, 2, ...N-1 \quad (13)$$

The more rigorous way of arriving at the above equation is to use the backward Kolmogorov equation (6) and let $t \to \infty$. In, this limit, $dP(N,m)/dt \to 0$ and the remaining of the equation reduces to eq.(12). We have $N-1$ unknown and equation, which we can write in matricial notation $AX = B$ as

$$\begin{pmatrix} 1 & -w_1^+ & & & \\ -w_2^- & 1 & -w_2^+ & & \\ & \ddots & \ddots & \ddots & \\ & & -w_{N-2}^- & 1 & -w_{N-2}^+ \\ & & & -w_{N-1}^- & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{N-2} \\ \pi_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w_{N-1}^+ \end{pmatrix}$$

Numerically of course, any linear package solver will handle this very easily. We can also solved it analytically: equation (12) may seem to be a two term recurrence equation ; it can however be trivially reduced to a one term recurrence equation by noting that $w^+(m)+w^-(m) = 1$. Setting $y_k = \pi_N(k+1) - \pi_N(k)$, equation (12) can be rewritten as

$$w^+(m)y_m - w^-(m)y_{m-1} = 0 \;\; m = 1, ..., N-1$$

which is easily solved into

$$y_m = f(m)y_0 \;\; m = 1, ..., N-1$$

The coefficient y_0 itself is determined by summing over all indexes ans noting that

$$\sum_{m=1}^{N-1} y_m = \pi_N(N) - \pi_N(0) = 1$$

and therefore

$$y_m = \frac{f(m)}{\sum_{m=1}^{N-1} f(m)}$$

which leads to

$$\pi_N(m) = \frac{\sum_{i=1}^{m-1} f(i)}{\sum_{i=1}^{N-1} f(i)}$$

IV. ABSORPTION MEAN TIME.

A. Bounded system, one absorbing states.

An other important concept is the fixation time, *i.e.* "mean time to fixation" or "first passage time". Let us again consider the bounded system (between $n = 0$ and $n = N$). Let us moreover suppose that state $n = N$ is absorbing:$W^-(N) = 0$. Each trajectory i beginning at state m will eventually reach the state N at some time t_i and remain there. We want to compute $T(m)$, the average over all these absorption times t_i (figure 3). The short answer is

$$T(m) = \int_0^\infty (1 - P(N, m; t))\, dt \quad (14)$$

The reason is the following : let $p(t; m)$ be the probability density of beginning at state m and getting absorb during

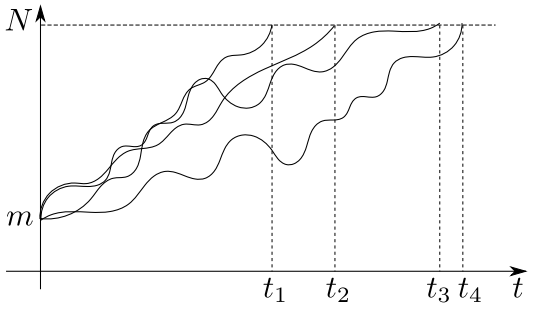

Figure 3: mean time to fixation

$[t, t+dt[$. Using again the tricks of "and"s, we have

$$p(t,m) = P(N-1,m;t)W^+(N-1)$$

As $W^-(N) = 0$, a short look at the boundary condition (8) shows that the right hand side of this equation is just $dP(N,m;t)/dt$ so the mean time to absorption is

$$\begin{aligned} T(m) &= \int_0^\infty tp(t,m)dt \\ &= \int_0^\infty t\left(dP(N,m;t)/dt\right)dt \\ &= [t(1-P(N,m;t)]_0^\infty + \int_0^\infty (1-P(N,m;t))\,dt \\ &= \int_0^\infty (1-P(N,m;t))\,dt \end{aligned}$$

Note that the condition $T(N) = 0$ is automatically satisfied. We can now use equation (14) to compute, with the help of the backward Kolmogorov equation (6) the quantity

$$W^-(m)[T(m-1)-T(m)] + W^+(m)[T(m+1)-T(m)] = \int_0^\infty -\frac{dP(N,m;t)}{dt}dt$$

Which leads to the equation for mean absorption time

$$W^-(m)[T(m-1)-T(m)] + W^+(m)[T(m+1)-T(m)] = -1 \qquad m = 0,1,...N-1 \quad (15)$$

In principle, care should be taken with the state $m = 0$ (see discussion on the boundary condition), but with bounded states, $W^-(0) = 0$ so we can use the above equation without writing explicitly the boundary condition. Again, the above equation can be solved numerically by resolving the matricial system

$$\begin{pmatrix} W_0^+ & -W_0^+ & & & \\ -W_1^- & \Sigma W_1 & -W_1^+ & & \\ & \ddots & \ddots & \ddots & \\ & & -W_{N-2}^- & \Sigma W_{N-2} & -W_{N-2}^+ \\ & & & -W_{N-1}^- & \Sigma W_{N-1} \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \\ \vdots \\ T_{N-2} \\ T_{N-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

Note the difference between the first and last line and compare them to the analog matrix for fixation probabilities.

We can also use the same argument as in the previous section to solve the system exactly. The solution is slightly more complicated because the recurrence equation is this time

$$W^+(m)y_m - W^-(m)y_{m-1} = -1$$

which is a non-homogeneous recurrence equation. Theoretically however, the solutions are obtained in the same way. Solving the equation for $y_m = T_{m+1} - T_m$ leads to

$$\begin{aligned} y_0 &= -1/W^+(0) \\ y_m &= f(m)y_0 + g(m) \end{aligned}$$

and T_0 for example is obtained by summing up the above equation.

B. Bounded system, two absorbing states.

Nothing really changes if the two states $n = 0$, $n = N$ are absorbing. Following the same line of arguments leads to

$$T(m) = \int_0^\infty (1 - P(N,m;t) - P(0,m;t)\,dt$$

which checks automatically $T(0) = T(N) = 0$. Using the backward Kolmogorov equation, we find again the same equation for the mean fixation time, except that the index varies from $m = 1, 2, ...N-1$.

C. Unbounded systems.

Many stochastic systems are unbounded, or semi-unbounded. The question is how long the system stays between two bounds, say $n = 0$ and $n = N$. This problem can be brought back to the previous problem by artificially setting $W^\pm(N) = 0$ and $W^\pm(0) = 0$, *i.e.* as soon as the system reaches one bound, we record the time at start again from the same initial state m.