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On the Eigenvalues of Orbital Angular Momentum

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Using only the elementary commutation relations in quantum mechanics, it is shown that the eigenvalues of $L_z = xp_y - yp_x$ are integers.

The eigenvalue problem for the orbital angular momentum operator

$$L = r \times p \tag{1}$$

has been one of the least satisfactorily discussed topics in elementary quantum mechanics. In the discussions found in most of the textbooks,[1] one usully starts from the commutation relations implied by (1) and derives the result that the eigenvalues of L_z can only be half-integers $(0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2} \cdots)$. This result is obtained by purely abstract considerations without any need for the use of fuction spaces. It is sufficient to simply use an abstract Hilbert space without demanding any specific realization of the space. At this point the problem of he elimination of the $\frac{1}{2}$ integral eigenvalues arises. This is usually done by gong outside the abstract Hilbert space framework and realizing (1) as an operator in a function space. Then with the help of some further restrictions, such as the single-valuedness requirement on the eigenfunction in the Schrodinger represntation,[1] one rules out the half-odd integral values $(\pm\frac{1}{2}, \pm\frac{3}{2}, \cdots)$. The necessity for inclusion of a physical constraint and the explicit use of a particular representation in the discussion of an eigenvalue problem has caused some uneasy feelings among physicists and has been a subject of considerable debate for many

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(1) See, for example, E. Merzbacher, Quantum *Mechanics* (John Wiley & Sons, Inc., New York, 1963), pp. 355 and 174.

years.[2-6] While the restriction can certainly be formulated in a variety of seemingly harmless statements,[7] it is nevertheless annoying to have the necessity for introducing such conditions. Several years ago Buchdahl[8] and Louck[9] gave independent derivations of the eigenvalues of L_z without using any requirement. While they both recognized the fact that the correct eigenvalues are implied by the particular form (1) of the orbital angular momentum operator, their arguments do not take the most elegant form. Besides being rather lengthy and quite indirect, their derivations involve the use of particular representations for the orbital angular momentum operator.[10] Shortly thereafter, Merzbacher[11] pointed out the connection between the two-dimensional harmonic oscillator and the angular momentum in three dimensions which provides, for the first time, a direct derivation of the correct eigenvalues.[12] However, as far as we know, this proof has never been adopted in any textbook of quantum mechanics, presumably because the ingenious trick involved is not an everyday tool familiar to all students.

We wish to present in this note another proof which seems to us to be more direct and simpler in structure and, therefore, more suitable for classroom presentations. The proof is abstract in structure depending only on the form of the operator L_z (i.e. that it is built in a specific way out of the operators $\underset{\sim}{r}$ and $\underset{\sim}{p}$) and on the fact we are (as in the general angular momentum theoy) working in an abstract Hilbert space.

First let us write L_z as

$$\begin{aligned} L_z &= xp_y - yp_x \\ &= C^+C - (A^+A + B^+B) \end{aligned} \tag{2}$$

where

$$A = \frac{1}{\sqrt{2}}(p_x - ix)$$

(2) W. Pauli, Helv. Phys. Acta **12**, 147 (1939).
(3) D. Bohm, ***Quantum*** Theory (Prentice-Hall, Inc., Englewood Cliffs. New Jersey, 1951), pp. 389-390.
(4) J. M. Blatt and V. F. Weisskopf, ***Theoretical Nuclear Physics*** (John Wiley & Sons, Inc., New York, 1952), pp. 783 and 787.
(5) E. Merzbacher, Am. J. Phys. 30, 237 (1962).
(6) M. L. Whipman, Am. J. Phys. **34**, 656 (1966).
(7) For example, the comparison with experiments is considered in Ref. 3. The condition of the absence of source and sink for the probability current is mentioned in Ref. 4 and discussed in detail in Ref. 6. References to other considerations can also be found in Ref. 6.
(8) H. Buchdahi, Am. J. Phys. **30**, 829 (1962).
(9) J. D. Louck, Am. J. Phys. 31, 378 (1963).
(10) An alternate derivation was also given by Louck (Ref. 9) for the operator (1) in the four-dimensional Cartesian space.
(11) E. Merzbacher, Am. J. Phys. 31, 549 (1963).
(12) There also exist group-theoretical arguments which lead to the correct result. See, for example, J. Schwinger in ***Quantum Theory of Angular Momentum,*** L.C. Biedenharn and H. Van Dam, Eds. (Academic Press Inc., New York, 1965) and J. M. Lévy-Leblond, Am. J. Phys. 35, 444 (1967).

$$B=\frac{1}{\sqrt{2}}(p_y-iy)$$

$$C=B+iA.$$

The following relations can then be readily established by using the commutation relations between $\boldsymbol{r}$ and $\boldsymbol{p}(=1)$.

$$[A, A^+]=1 \quad (3)$$

$$[B, B^+]=1 \quad (4)$$

$$[C, C^+]=2 \quad (5)$$

$$[A^+A, B^+B]=0 \quad (6)$$

$$[C^+C, A^+A+B^+B]=0. \quad (7)$$

The proof is based on the following results well-known to all students of quantum mechanics[13] which we now present as two lemmas. If A and B are two operators in a Hilbert space, then;

Lemma 1.

The commutation relation $[A, A^+]=\lambda$ implies that the eigenvalues of A^+A are $0, \lambda, 2\lambda, 3\lambda, \ldots.$

Lemma 2.

If A, B commute, then the eigenvalues of A + B (or A-B! are some sums (or differences) of the eigenvalues of A and B.

From (3), (4) and Lemma 1, the eigenvalues of A^+A and B^+B are 0, 1, 2,. ... Hence by (6) and Lemma 2, the eigenalues of A^+A+B^+B are 0. 1,. 2, ... Similarly from (5) and Lemma 1, the eigenvalues of C^+C are 0, 2, 4, ... Hence from (7), Lemma 2, and the eigenvalues of A^+A+B^+B just deduced, the eigenvalves of L_z can only have positive of negative integral values including zero. This completes the proof.[14]

To summarize, we have shown that the operator L_z defined in a Hilbert space has integral eigenvalues only. The proof does not use any additional condition usually needed in the Schrodinger representation.

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(13) Lemma 1 is proved in almost any elementary textbook in quantum mechanics. See, for example, pp. 349-351 of Ref. 1. Lemma 2 follows from the fact that commuting operators have simultaneous eigenvectors.

(14) Technically speaking, our proof only rules out the non-integral eigenvalues. But this is the desired result.