Linear differential equations and solvability conditions.[*]

Bahram Houchmandzadeh, Grenoble Universités.

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1 Introduction.

The general problem we are concerned with in this lecture is the existence and unicity of solutions of a linear differential equation. For example, does the equation

$$\begin{aligned} y'' + \omega^2 y &= \sin t \\ y(0) = y'(0) = y(1) = y'(1) &= 0 \end{aligned}$$

has *a* solution ? And if yes, how unique is this solution ? The answer to this particular question is straightforward, because we can explicitly construct the answer. Very often however, we don't know the exact analytical answer, but we are very interested to know if it can exist. For example, the dynamics of an order parameter (such as the local magnetism at temperatures below the curie temperature) is given by the landau equation[1]

$$\frac{\partial u}{\partial t} = u - u^3 + \frac{\partial^2 u}{\partial x^2} + \epsilon$$

where ϵ is the external applied field. At $\epsilon = 0$, the above equation has a stationary solution

$$u_0(x) = \tanh x$$

In the presence of the field $\epsilon \neq 0$, we can look for a perturbation solution to the first order $u(x,t) = u_0(x) + \epsilon u_1(x,t)$ [complete later]

[*]Lanczos has written a book in the 60's on linear differential operators which I find very thoughtful. The following notes are a short introduction to part of his book. I have slightly modernized his vocabulary, specially because he used heavily the matrix formalism and I find personally that the view is more beautiful if we stay at the level of operators without explicitly writing down their matrix representation.

[1]which is just the least order expansion of the free energy as a function of the order parameter

2 Solvability of an algebraic linear system.

Let us suppose that we have m linear equations for m unknowns. We can shorten the notation by writing

$$Ax = b \tag{1}$$

where A is an $m \times m$ matrix, x an $m-$dimensional vector of unknown, and b an $m-$dimensional vector of known quantities. We can operate one more layer of abstraction : Let us think of A as a linear operator sending a vector of a vectorial $m-$dimensional space $\mathcal{E}$into another one ; moreover, let us suppose that we have equipped $\mathcal{E}$ with a scalar product $\langle x|y\rangle$. For our purpose here, this could be just the usual product $\langle x|y\rangle = \sum_i x_i y_i$ where x_i and y_i are the component of x, y in a given basis. But the detail of the scalar product is not important, what we want is to have *a* scalar product defined over the space $\mathcal{E}$. Now, we define the adjoint operator $A^\dagger$ operator such as

$$\left\langle A^\dagger x|y\right\rangle \stackrel{\text{def}}{=} \langle x|Ay\rangle$$

Practically speaking, the matrix of $A^\dagger$ is given by exchanging row and columns of the matrix of A. But let us not forget that a matrix is just a *representation* of an operator, its picture. The adjoint operator plays a fundamental role in linear operator theory.

The Kernel of an operator A, called $ker(A)$ is the set of all vector u such that $Au = 0$. It can easily be shown that $ker(A)$ is a subspace of $\mathcal{E}$, generated by eigen vectors associated to the 0 eigen values. With all these definitions in hand, we are know in a position to answer to existence and unicity question.

1. Existence. Let suppose that $u \in ker(A^\dagger)$. We can take the scalar product of u with the right and left hand of (1)

$$\langle u|Ax\rangle = \left\langle A^\dagger u|x\right\rangle = 0 = \langle u|b\rangle$$

So, the right hand side has to be orthogonal to every vector in $Ker(A)$. System (1) has a solution only if

$$b \perp ker(A^\dagger)$$

and so the kernel of the adjoint operator provides the existence condition.

2. Unicity. Let us suppose that x_1 and x_2 are solution of (1). Then, $A(x_1 - x_2) = 0$, which means that

$$x_1 - x_2 \in ker(A)$$

The solution of the linear system is therefore of the form $x + u$, where x is a particular solution and $u\in ker(A)$. The kernel of the operator provides the unicity.

An important case appear for symmetric systems[2] where $A^\dagger = A$; then the operator provides both for existence and unicity.

[2]we are only concerned here by vector spaces defined over reals. Generalization to complexes is trivial

Rectangular systems. An important generalization is the case for rectangular matrices. If we have n equation and m unknowns, what are the existence and unicity conditions ? We can again write our system as

$$Ax = b \tag{2}$$

but this time, we suppose that we have *two* vectorial spaces $\mathcal{E}_1$(m-dimension) and $\mathcal{E}_2$($n-$dimension) with $A : \mathcal{E}_1 \to \mathcal{E}_2$. The operator A has a $n \times m$ matrix representation. Again, we suppose that both spaces come equipped with an inner product and we define once again the adjoint operator by

$$\left\langle A^\dagger x | y \right\rangle \stackrel{\text{def}}{=} \langle x | Ay \rangle$$

Note however that the left hand side is a scalar product in $\mathcal{E}_1$ and the right hand side a scalar product in $\mathcal{E}_2$! practically, $A^\dagger$'s matrix is obtained by exchanging rows and columns of A and is therefore an $m \times n$ matrix.

We can now repeat every word of our above discussion.

- System (2) has a solution if $b \in ker(A^\dagger)$.
- The general solution of (2) is written $x + u$, where x is a particular solution and $u \in ker(A)$.

3 Hilbert Space.

Functions can be considered as vectors, and the function space a vector space. We know how to define addition, multiplication by a scalar, we have a 0 vector, ... If we restrict ourselves to square integrable functions over an interval $[a, b]$ (which can be infinite), we even have a scalar product :

$$\langle f | g \rangle = \int_a^b f(x) g(x) dx$$

Solving a differential equation $f''(x) = \beta(x)$ can be written in operatorial notation as

$$Af = \beta$$

where A refers to the second derivation operator $Af = f''$. Whatever we said for linear systems can be repeated here (even though the space is of infinite dimension) *if* we pay close attention to boundary conditions. And this is a big if.

A differential equation without boundary conditions is like a rectangular system. This can be seen by writing down a discrete version of the differential equation over a grid with N points ; doing that, we realize that there is N unknown, but only $N-r$ equations, where r is the degree of the equation. So a differential equation without boundary conditions is undetermined and its solution belongs to a subspace of the Hilbert space. For example, $y'' + \omega^2 y = 0$ will have as solution $y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$: the solution is *a* linear combination of the two vectors $\cos(\omega t)$ and $\sin(\omega t)$, or worded differently, belongs to the subspace generated by these two vectors. It's only when we provide at least two boundary conditions that a unique solution can be specified.

3.1 Homogeneous boundary conditions.

When dealing with differential equations, we have to consider a differential operator *and* a certain amount of boundary conditions. Now, consider the differential equation $A.y(x) = h(x)$ (where A is a differential operator) with the boundary condition $y(0) = 1$. We can see it as the operator A acting on the space of functions with this particular boundary conditions and try to repeat all the arguments we had on linear operator. We get into a serious problem however : the space $\mathcal{F}$ of functions $f(x)$ with the condition $f(0) = 1$ is *not* a vectorial space. Specifically, if $f, g \in \mathcal{F}$, then $f + g \notin \mathcal{F}$, because $f + g$ will have a boundary condition equal to 2[3].

Let us now consider the same equation $A.y(x) = h(x)$, but this time with the boundary condition $y(0) = 0$. We can be happy again, because the space $\mathcal{F}$ of functions $f(x)$ with the condition 0 is a vectorial space. From now on, we will consider homogeneous (*i.e.* $= 0$) boundary conditions. We will come back later to the general boundary conditions.

We can repeat then whatever we said in previous sections on linear operators and kernels. Let us first define the adjoint operator *and* its associated boundary condition. By our previous definition,

$$(f, Ag) \stackrel{\text{def}}{=} (A^\dagger f, g)$$

we will ask the minimum boundary conditions for the adjoint operator in order to satisfy the above equality. This will define on which space the adjoint operator acts.

Example 1. Let us consider the operator $D = d/dx$ with no boundary condition, acting on the space $\mathcal{F}_1$ of (sufficiently smooth) functions define on the interval $[0, 1]$. We have to find the operator $A^\dagger$ and the space $\mathcal{F}_2$ it acts upon, by using the definition. Let's suppose that $g \in \mathcal{F}_1$ and $f \in \mathcal{F}_2$ are two arbitrary function. Then

$$\int_0^1 f(x).\frac{d}{dx}g(x).dx \quad = \quad [f(x)g(x)]_0^1 - \int_0^1 \left(\frac{d}{dx}f(x)\right).g(x).dx$$

We first of all note that the adjoint operator is $A^\dagger = -d/dx$. Moreover, because $\mathcal{F}_1$ is free of boundary conditions, we have to choose $\mathcal{F}_2$ as the space of functions with boundary conditions $f(0) = f(1) = 0$.

Let us know consider the differential equation $Ag(x) = \eta(x)$ where $\eta(x)$ is a known function. Does this equation has a solution ? Is it unique ? We know that the existence is given by the condition $\eta(x) \perp ker(A^\dagger)$. The equation $A^\dagger f(x) = 0$ with the specified boundary conditions has no other solution than $f(x) = 0$; therefore, the existence condition is always fulfilled. The unicity is provided by $ker(A)$ which includes all functions $g(x) = \text{Cte}$; therefore, the solution of $Ag = \eta(x)$ exists and is defined up to a constant.

Example 2. Study the solvability of the equation

$$y'' + y = \eta(x)$$

[3] To be precise the function space with general boundary conditions is an affine space, a space where we have lost the origin. The difference of two elements of an affine space belongs to a vectorial space.

with the boundary conditions $y(0) = y'(0) = y(1) = y'(1) = 0$.

It is not difficult to see that the adjoint operator associated to $A = d^2/dx^2 + 1$with the above boundary conditions is $A^\dagger = d^2/dx^2 + 1$ with *no* boundary conditions. The equation $A^\dagger f(x) = 0$ has two independent solution $\cos(x)$ and $\sin(x)$ and the existence condition $\eta(x) \perp ker(A^\dagger)$ yields

$$\begin{aligned} \int_0^1 \eta(x)\sin(x)dx &= 0 \\ \int_0^1 \eta(x)\cos(x)dx &= 0 \end{aligned}$$

To rephrase it in physicist language : an harmonic oscillator at rest at time $t = 0$ submitted to a force $\eta(t)$ will be back to its original condition at time $t = 1$ iff the force η fulfills the above two conditions.

The unicity condition on the other hand is fulfilled.

3.2 General boundary conditions.

Existence. We want to investigate the solvability of the equation

$$Ag(x) = \eta(x) \tag{3}$$

with boundary conditions $g^{(k)}(x_i) = \alpha_{ki}$. Let us consider the same equation, with the same boundary conditions but homogeneous : $g^{(k)}(x_i) = 0$; then we can define as usual the adjoint operator (with its specified boundary condition) and its kernel $ker(A^\dagger_{\text{homogen}})$. Now, Let us take the scalar product of eq.(3) with $u(x) \in ker(A^\dagger_{\text{homogen}})$. Then,

$$\langle u(x)|\eta(x)\rangle = \langle u(x)|Ag(x)\rangle = \text{boundary terms} \tag{4}$$

The scalar product of $u(x)$ and $Ag(x)$ is not zero, because A here possess all the real boundary terms. The scalar product however will now be defined by these known boundary conditions. Equation (4) is the new solvability condition. Let us look at a simple example to set the idea.

Consider the equation $y' = \eta(x)$ with boundary conditions $y(0) = a; y(1) = b$. The homogeneous adjoint operator is $-d/dx$ with no boundary conditions and therefore, its kernel contains constant functions $f(x) = C$. We will no take the scalar product of elements of the kernel, mainly the function $f(x) = 1$ with both side of eq(3), having in mind that operator A acts on functions with the real prescribed boundary conditions.

$$\begin{aligned} \langle 1|\eta(x)\rangle &= \int_0^1 \eta(x) \\ &= \langle 1|Ag(x)\rangle \\ &= \int_0^1 1.\left(\frac{d}{dx}g(x)\right)dx \\ &= g(1) - g(0) \end{aligned}$$

This differential equation is solvable only if

$$\int_0^1 \eta(x)dx = b - a$$

Unicity. what we did above can be repeated for the unicity. Consider the operator A_{homogen} . Any solution of eq(3) is defined up to a function $v(x) \in ker(A_{\text{homogen}})$. The demonstration is trivial.