Master de Sciences de la Matière, deuxième année
année 2007-2008

NONEQUILIBRIUM STATISTICAL PHYSICS
Première Session, 9 Novembre 2007, durée 3h

1 Qualitative questions

Answer qualitatively each question based on your understanding of the lectures. You do not need to carry out any calculation, but give precise qualitative arguments.

1. Is the following statement always true ? If not, give restrictive conditions for its validity.
"I*n a system out of equilibrium, described by a Langevin type equation, the intensity of the noise is proportional to temperature*"

2. Suppose a system has a structure factor $S(k)$ of the form
$$S(k) = \frac{A}{B(T - T_c) + (k - k_0)^2}$$
What kind of phase transition do you expect when T approaches T_c ? What is the correlation length ? What is the consequence for the dynamics of the fluctuations described by $S(k)$, in the vicinity of k_0?

3. How many conserved densities are present in a mixture of two fluid components? In addition to the heat diffusion and sound wave modes, which mode do you expect to be present in a dynamic light scattering measurement of $S(k, \omega)$?

2 Langevin equation with memory

The assumption of a random force that has strictly no correlations in time, and that the friction is local in time, are simplifying assumptions that characterize the original Langevin model. More generally, it is possible to consider a Langevin type equation that involves a nonlocal (in time) friction term and a random force that is not delta correlated. For a particle in one dimension, the equation is:

$$M\dot{v} = -\int_{-\infty}^{t} M\zeta(t - s)v(s)ds + F_{ext}(t) + \theta(t). \tag{1}$$

with

$$\langle \theta(t)\theta(t') \rangle = \Theta_0(t - t'). \tag{2}$$

The function $\zeta(t)$ is called the memory function, and $\Theta_0(t)$ characterizes the power spectrum of the noise $\theta(t)$. $F_{ext}(t)$ is an external force applied to the particle As in the usual Langevin equation, $\langle \theta(t) \rangle = 0$. $\zeta(t) = 0$ if $t < 0$.

1. For an oscillatory external force $F_0 \exp(i\omega t)$, define the frequency dependent mobility $\lambda(\omega)$. Use the fluctuation-dissipation theorem to show that, for any system at thermal

equilibrium, the frequency dependent mobility $\lambda(\omega)$ is related to the autocorrelation of the velocity $Z(t) = \langle v(t)v(0)\rangle$ through

$$Z(\omega) = \int_{-\infty}^{+\infty} Z(t)e^{i\omega t}dt = 2k_BT\mathrm{Re}[\lambda(\omega)]$$

2. Use the Langevin equation **??** without external force to show that $Z(\omega)$ (Fourier transform of $Z(t)$) is related to the Fourier transform $\Theta_0(\omega)$ of $\Theta_0(t)$ by

$$Z(\omega) = \frac{\Theta_0(\omega)}{M^2(\omega^2 + |\zeta(\omega)|^2)}$$

3. Use the Langevin equation with an oscillatory external force to relate the frequency dependent mobility $\lambda(\omega)$ and $\zeta(\omega)$

4. Combine the results obtained in 1, 2, and 3 to show that $\Theta_0(\omega)$ and the memory function $\zeta(\omega)$ are related by

$$\mathrm{Re}[\zeta(\omega)] = \frac{1}{2Mk_BT}\Theta_0(\omega)$$

What is the corresponding equality in the time (rather than frequency) domain?

5. We now specialize to the case where the memory function $\zeta(t)$ is exponential, $M\zeta(t) = K\exp(-t/\tau)$. Explain why the product $K\tau$ is related to the viscosity of the fluid, and in what sense K can be considered to be related to a high frequency elastic modulus.

6. Show that the Laplace transform of the velocity autocorrelation function is of the form

$$\tilde{Z}(z) = \frac{k_BT}{M(z + \tilde{\zeta}(z))}$$

Hint: you will need to use the results of question 1, and the equalities $\tilde{Z}(z) = \frac{1}{2}\mathbf{Re}(Z(\omega = iz))$ (use the fact that $Z(t) = Z(-t)$) and $\tilde{\zeta}(z) = \zeta(\omega = iz)$.

7. Show that the velocity autocorrelation function can be written as

$$Z(t) = A_1\exp(z_1t) + A_2\exp(z_2t)$$

Express z_1 and z_2 as a function of the parameters K, τ and give an interpretation of this dependence.

Express A_1 and A_2 as a function of z_1, z_2 and k_BT/M (use the initial values of Z and $\dot{Z}$).

8. Show that the noise $\theta(t)$ can be obtained as the solution of a simple Langevin equation of the form

$$\dot{\theta}(t) = -K_1\theta(t) + R_1(t)$$

Give the value of K_1 and the statistical properties of R_1.

9. By introducing a new variable $u(t) = -M\int_{-\infty}^{t}\zeta(t-s)\mathbf{v}(s)ds + \theta(t)$, show that equation **??** can be rewritten in the form of two coupled Langevin equations without memory

$$\dot{v} = u(t)/M$$

$$\dot{u} = -F(u, v) + R_1(t)$$

Specify the function $F(u, v)$.

10. Write the Fokker-Planck equation for the distribution associated with the two coupled Langevin equations above.