Ecole Normale Supérieure de Lyon *Université Claude Bernard Lyon I*

Master de Sciences de la Matière, deuxième année
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NONEQUILIBRIUM STATISTICAL PHYSICS
Première Session, 9 Novembre 2008, durée 3h

1 Properties of the noise in fluctuating hydrodynamics

Fluctuating hydrodynamics describes long wavelength fluctuations in a fluid using a Langevin approach applied to the velocity field. The equation for the transverse part of the velocity field is written in the form

$$\rho \frac{\partial \overrightarrow{\mathbf{v}}_\perp(\vec{r},t)}{\partial t} = \eta \nabla^2 \overrightarrow{\mathbf{v}}_\perp(\vec{r},t) + \overrightarrow{\Theta}(\vec{r},t) \tag{1}$$

where $\Theta(\vec{r},t)$ is a stochastic noise field. $\overrightarrow{\Theta}(\vec{r},t)$ is a vector.

1. Show that the noise field must obey

$$\nabla \cdot \overrightarrow{\Theta}(\vec{r},t) = 0 \quad \langle \overrightarrow{\Theta}(\vec{r},t) \rangle = \overrightarrow{0} \tag{2}$$

$$\langle \Theta_i(\vec{r},t)\Theta_j(\vec{r}',t') \rangle = 2k_B T \eta \delta(t-t')\delta_{ij}\nabla^2\delta(\vec{r}-\vec{r}') \tag{3}$$

   You will need to use the Fourier space representation of $\overrightarrow{\Theta}(\vec{r},t)$

2. Suppose that one wants to perform a numerical implementation of equation 1. The fields will be discretized on a lattice with periodic boundary conditions. Assume that you have at your disposal two programs, one that gives independent random numbers (e.g. a uniform distribution between -1 and $+1$), and one that performs discrete Fourier transforms. How would you proceed to generate a realisation of the noise $\overrightarrow{\Theta}(\vec{r},t)$, obeying the conditions above, in real space ?

2 Fluctuations of a galvanometer frame

At a macroscopic level, the motion of a galvanometer frame of resistance R and inductance L can be described by two coupled differential equations, a mechanical equation for the rotation angle α and an electrical equation for the intensity $i(t)$ in the electrical circuit..
Mechanical equation:

$$J\frac{d^2\alpha}{dt^2} = -f\frac{d\alpha}{dt} - C\alpha(t) - Ki(t) \tag{4}$$

In this equation, J is the moment of inertia, $-f\frac{d\alpha}{dt}$ is the torque that results from the friction of the surrounding fluid, $-C\alpha$ is the torque exerted by the torsion spring, and $-Ki(t)$ is a torque of magnetic origin.
Electrical equation:

$$L\frac{di}{dt} = -Ri(t) + K\frac{d\alpha}{dt} + e(t) \tag{5}$$

The second term in the right hand side is the electromotive force of magnetic origin, $e(t)$ is the electromotive force of a generator. In the following we will assume that the galvanometer is in closed circuit with $e(t) = 0$.

1. Write the energy of the system as a function of $i(t)$, $\frac{d\alpha}{dt}$, $\alpha(t)$.

2. Based on energy conservation arguments, explain why the same constant K appears in the two equations 4 and 5.

3. In order to describe thermal fluctuations, the macroscopic equations are interpreted as Langevin equations.

   A random torque $\Gamma(t)$ is introduced in equation 4, and a random electromotive force $E(t)$ is introduced in equation 5. These electrical and mechanical noises have a zero average value and correlations of the Langevin type:

   $$\langle \Gamma(t)\Gamma(t')\rangle = 2\Gamma_0\delta(t-t'); \quad \langle E(t)E(t')\rangle = 2E_0\delta(t-t'); \quad \langle E(t)\Gamma(t')\rangle = 0 \qquad (6)$$

   Write the corresponding Langevin equations and relate the intensity of the noise terms to the friction terms appearing in each equation.

4. In analogy with the study of fluctuations in fluids, show that the correlation functions $< \alpha(t)\alpha(0) >$, $< I(t)I(0) >$ can always be written as sums of three exponentials, e.g.:

   $$\langle \alpha(t)\alpha(0)\rangle = C_1 e^{z_1 t} + C_2 e^{z_2 t} + C_3 e^{z_3 t} \qquad (7)$$

   where z_1, z_2, z_3 are the roots of the cubic equation

   $$Jz^2(Lz+R) + (fz+C)(Lz+R) + K^2 z = 0 \qquad (8)$$

5. One now assumes that the inductance is small, $L = 0$. Write the stochastic equation obeyed by α. Show that the probability distribution function $P(\alpha, t)$ obeys a Kramers equation.

6. What does "small inductance" actually mean ? To answer this question, consider the three roots of equation 8 in the limit of weak coupling (small K).

7. Keeping $L = 0$, we now consider the case where the electrical part of the system is at a temperature T' while the surrounding fluid is at a temperature T. Show that the variable α will behave as if it was coupled to a thermal bath at some intermediate temperature T_e. Compute T_e as a function of T, T, f,K, and R.

8. Assume now that both the inductance L **and the torsion spring** C **can be ignored. The equation of motion for the angular velocity** $\omega = \frac{d\alpha}{dt}$ **is now of the general form**

   $$J\frac{d\omega}{dt} = -f\omega - f'\omega + \Gamma(t) + \Gamma'(t)$$

   where $\Gamma(t)$ **and** f **are the random torque and the friction associated with the fluid at temperature** T**, and** $\Gamma'(t)$ **and** f' **are the torque and friction coming from the coupling with the electrical circuit at temperature** T'**. The work per unit time done by the fluid on the galvanometer frame is**

   $$\mathcal{P} = \langle(-f\omega(t) + \Gamma(t))\omega(t)\rangle$$

   Using the general solution of the Langevin equation for $\omega(t)$**, compute** $\mathcal{P}$ **as a function of** J, T, T', f **and** f'**. Discuss the result.**