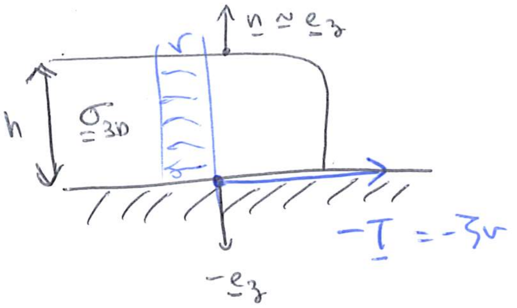
Contractility and depolymerization in an elastic environment

I) Kinematics & 1D

① From 3D to 1D: traction forces



$$\underline{\underline{\sigma}}_{3D} \cdot \underline{e}_z \Big|_{z=h} = \underline{0} = \begin{pmatrix} \sigma^{3D}_{xz} \\ \sigma^{3D}_{yz} \\ \sigma^{3D}_{zz} \end{pmatrix}_{z=h}$$

$$\underline{\underline{\sigma}}_{3D} \cdot (-e_z) \Big|_{z=0} = \begin{pmatrix} -T_x \\ -T_y \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma^{3D}_{xz} \\ \sigma^{3D}_{yy} \\ \sigma^{3D}_{zz} \end{pmatrix}_{z=0}$$

$$h\,\underline{\underline{\sigma}}_{3D} = \begin{pmatrix} \boxed{\underline{\underline{\sigma}}^{(2D)}} & \begin{matrix} -(h-z)T_x \\ -(h-z)T_y \end{matrix} \\ \begin{matrix} -(h-z)T_x & -(h-z)T_y \end{matrix} & 0 \end{pmatrix} \;..$$

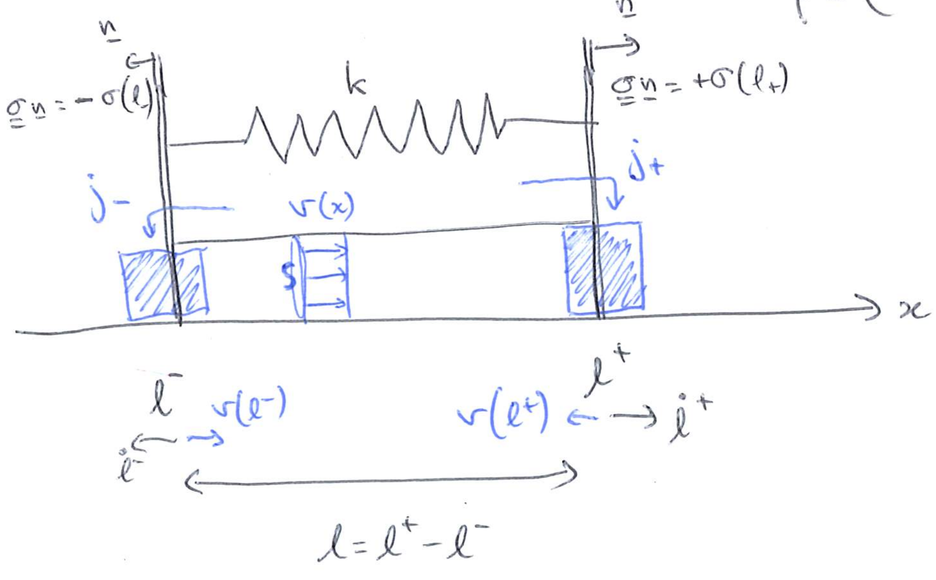
$$\underline{0} = \underline{\nabla}_{3D} \cdot \underline{\underline{\sigma}}^{3D} \Rightarrow \underline{\nabla} \cdot \underline{\underline{\sigma}}^{(2D)} = T = \zeta v$$

② 1-dimensional balance of a contractile segment with boundary growth

$$\partial_x \sigma = \zeta v$$

$$\pm\sigma(l_\pm) = \mp \frac{k}{S}(l - l_0)$$

$$\rho S\left(\dot{l}_\pm - v(l_\pm)\right) = \pm j_\pm$$



What does it model?
- cell crawling
- dividing cell tether

II) Contractile gel with boundary polymerisation

Constitutive eqn: $\sigma = \eta \partial_x v + \underbrace{\lambda_{12} \Delta\mu_c}_{\text{assumed uniform}}$

Insert into the mechanical balance eqn:

$$\partial_x \sigma = \eta \partial^2_{xx} v + 0 = \xi v$$

Solutions have the shape:

$$v = v_0 \sinh\left(\sqrt{\frac{\xi}{\eta}}\,(x - x_c)\right)$$

$$\sigma = v_0 \sqrt{\eta\xi}\, \underline{\cosh\left(\sqrt{\frac{\xi}{\eta}}\,(x - x_c)\right)} + \lambda_{12}\Delta\mu_c$$

Boundary conditions:

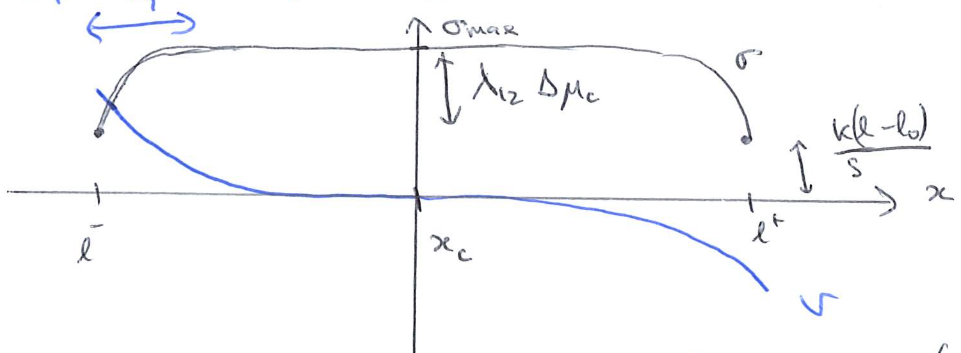
$$0 = \sigma(\ell^+) - \sigma(\ell^-) = v_0\sqrt{\eta\xi}\left(\cosh\left(\sqrt{\frac{\xi}{\eta}}(\ell^+ - x_c)\right) - \cosh\left(\sqrt{\frac{\xi}{\eta}}(\ell^- - x_c)\right)\right)$$

$$\Rightarrow x_c = \frac{\ell_- + \ell_+}{2}$$

$$-\frac{k}{S}(\ell - \ell_0) = \sigma(\ell^+) + \sigma(\ell^-) = 2 v_0 \sqrt{\eta\xi}\cosh\left(\sqrt{\frac{\xi}{\eta}}\,\frac{\ell}{2}\right) + 2\lambda_{12}\Delta\mu_c$$

$$\Rightarrow v_0 = -\frac{\frac{k(\ell-\ell_0)}{S} + \lambda_{12}\Delta\mu_c}{\sqrt{\eta\xi}\cosh\left(\sqrt{\frac{\xi}{\eta}}\,\frac{\ell}{2}\right)}$$

$L_F = \sqrt{\frac{\eta}{\xi}}$ hydrodynamic length



$$v(\ell_\pm) = \mp \frac{\sigma_{max}}{\sqrt{\eta\xi}} \tanh\left(\frac{\ell}{2L_F}\right)$$

Time evolution

- of the length:

$$\dot{\ell} = \dot{\ell}_+ - \dot{\ell}_- = \frac{j_+ + j_-}{\rho_p S} - \frac{k(\ell - \ell_0) + \lambda_{12}\Delta\mu_c}{\sqrt{\eta\zeta}} \tanh\left(\frac{\ell}{2L_f}\right)$$

<u>Steady state $\dot{\ell} = 0$,</u>

For $L_f \gg \ell$: $\quad \ell^* = \ell_0 - \dfrac{\lambda_{12}\Delta\mu_c}{k} + \dfrac{\sqrt{\eta\zeta}\,(j_+ + j_-)}{\rho_p S k}$.

For $L_f \ll \ell$: $\quad \ell^* = \ell_0 - \dfrac{\lambda_{12}\Delta\mu_c}{k} + \dfrac{4\sqrt{\eta\zeta}\, L_f\, (j_+ + j_-)}{\rho_p S (k\ell_0 - 2\sigma_a)}$
large k

for $k = 0$,
$$\begin{cases} \ell^* = 2L_f \operatorname{arctanh} \dfrac{\sqrt{\eta\zeta}\,(j_+ + j_-)}{\rho_p S\, \lambda_{12}\Delta\mu_c} \rightarrow \dfrac{\eta\,(j_+ + j_-)}{\rho_p S \lambda_{12}\Delta\mu_c} \text{ as } \zeta \to 0 \\ \text{or} \\ \text{unbounded growth if } \underbrace{\qquad\qquad}_{\geq 1} \end{cases}$$

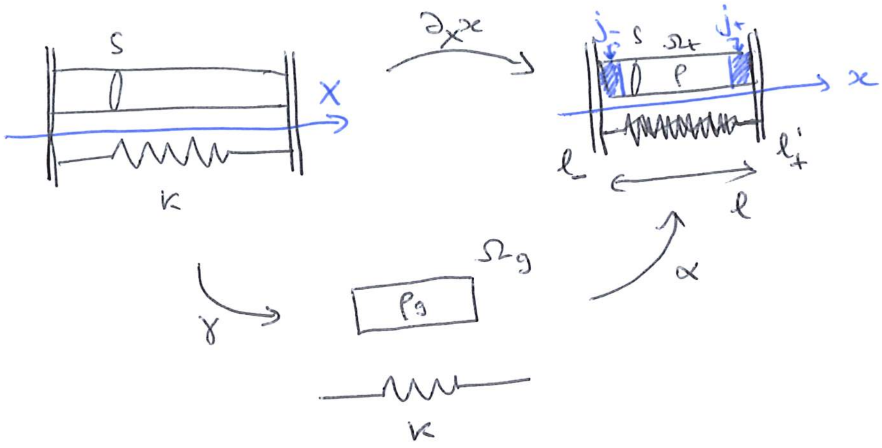
- of the position:

$$\dot{x}_c = \frac{1}{2}\left(\dot{\ell}_+ + \dot{\ell}_-\right) = \frac{j_+ - j_-}{\rho_p S}$$

friction independent! ... put by hand.
$\dot{x}_c \neq 0$ can also result from an instability; if $j_+ = j_-$

but what about $\rho(t)$?

III) Shrinking by depolymerisation of an elastic bar



$$\partial_X x = \alpha\gamma$$
$$\partial_X v = \dot\alpha\gamma + \alpha\dot\gamma$$
$$\partial_x v = \frac{\partial v}{\partial X}\frac{\partial X}{\partial x} = \frac{\dot\alpha}{\alpha} + \frac{\dot\gamma}{\gamma}$$

ρ_g is fixed, $\rho(t) = \dfrac{\rho_g}{\alpha(t)}$

No friction for simplicity, $\xi = 0$ (compare $L_f \to +\infty$)

Constitutive choices

$$f(\alpha) = \bar{f} + f_0(\alpha \ln\alpha - \alpha) \qquad \text{as in lecture 4}$$

Purely elastic

$$\sigma = \rho_g f_0 \ln\alpha$$

Growth law:

$$\rho\frac{\dot\gamma}{\gamma} = \lambda(\mu_0 - \bar{f} + f_0\alpha)$$

$$\Rightarrow \rho_g\frac{\dot\gamma}{\gamma} = \lambda(\mu_0 - \bar{f} + f_0\alpha)\alpha$$

Balance equations

$$\partial_x\sigma = 0 \qquad\qquad \partial_t\rho + \partial_x(\rho v) = \rho\frac{\dot\gamma}{\gamma}$$

$$\pm\sigma\big|_{l_\pm} = \mp\frac{k}{S}(l - l_0) \qquad\qquad \rho S(\dot{l}_\pm - v(l_\pm)) = \pm j_\pm$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sigma \equiv -\frac{k(l(t) - l_0)}{S} \qquad\qquad \rho_g S\,\alpha^{-1}(\dot{l}_\pm - v(l_\pm)) = \pm j_\pm$$

$$\alpha \equiv e^{-\frac{k(l(t) - l_0)}{S\rho_g f_0}}$$

γ and $\partial_x v$ also uniform

Subtract left and right BC for an equation of length evolution:

$$\rho_g S \alpha^{-1} \Big(\underbrace{\dot{\ell}_+ - \dot{\ell}_-}_{\dot{\ell}} - \underbrace{v(\ell_+) + v(\ell_-)}_{-\int_{\ell^-}^{\ell^+} \partial_x v \, dx} \Big) = j_+ + j_-$$

Since $\partial_x v$ is independent of x,

$$\rho_g S \alpha^{-1} \left(\dot{\ell} - \ell \, \partial_x v \right) = j_+ + j_-$$

Look for a permanent regime:

$$\dot{\ell} = 0 \Rightarrow \dot{\sigma} = 0 \Rightarrow \dot{\alpha} = 0 \Rightarrow \partial_x v = \frac{\dot{\gamma}}{\gamma}$$

Using constitutive equation for $\dot{\gamma}/\gamma$:

$$-S \cancel{\alpha^{-1}} \, \ell^* \times \lambda(\mu_0 - \bar{f} + f_0 \alpha) \cancel{\alpha} = j_+ + j_-$$

$$\ell^* \left(\exp\left(- \frac{k(\ell^* - l_0)}{S\rho_g f_0} \right) + \frac{\mu_0 - \bar{f}}{f_0} \right) = - \frac{j_+ + j_-}{\lambda S f_0}$$

ℓ^* exists only if $\mu_0 - \bar{f} < 0$, else unbounded growth

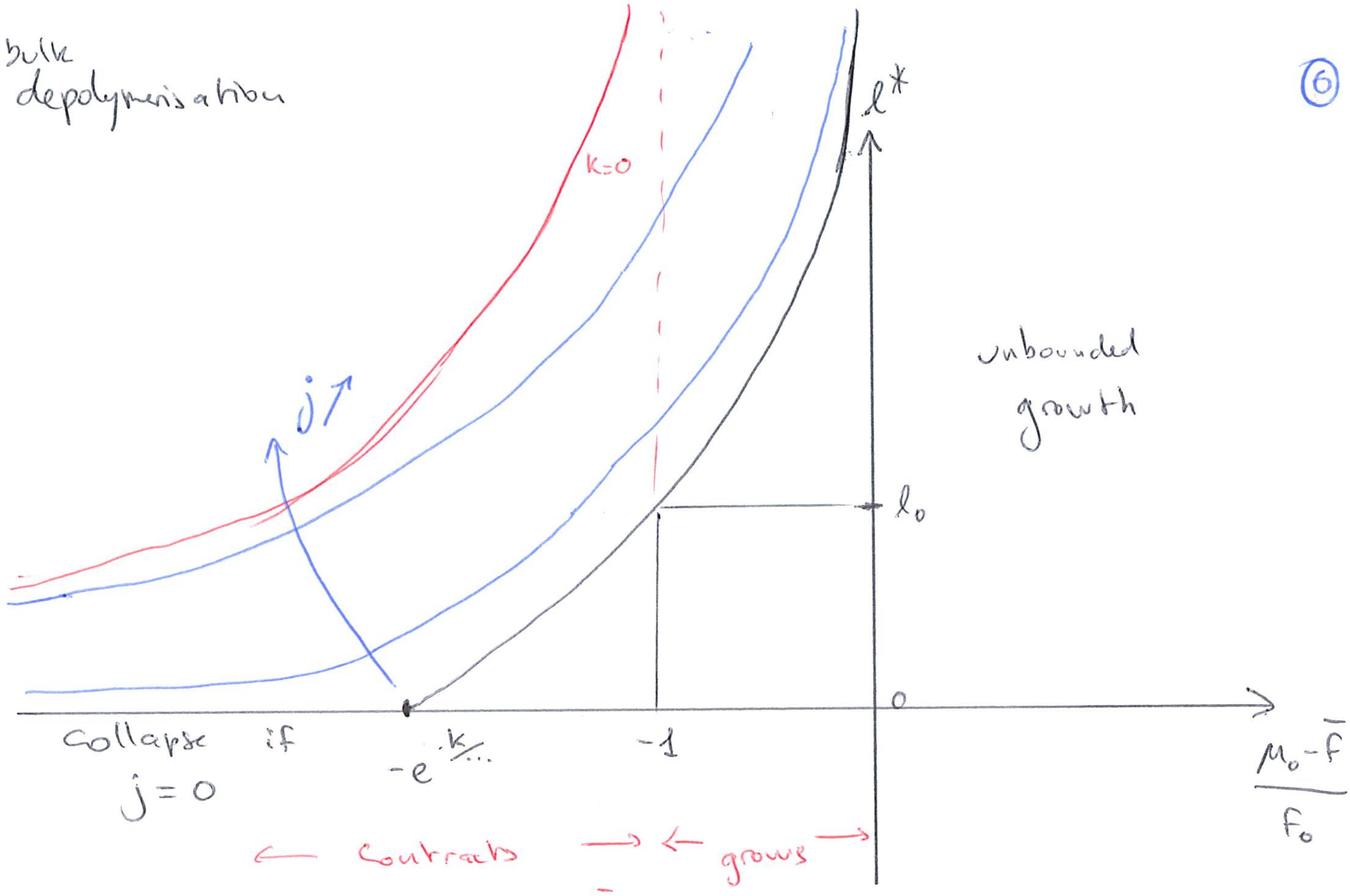
For k sufficiently large, $j_+ + j_-$ small:

$$\ell^* \simeq l_0 + \underbrace{\frac{S\rho_g f_0}{k} \log \frac{f_0}{\bar{f} - \mu_0}}_{\text{can be } < 0} + \frac{(j_+ + j_-)}{\lambda(\bar{f} - \mu_0)\Big(\underbrace{S f_0 \log\left(\frac{f_0}{\bar{f} - \mu_0}\right)}_{\lesssim 0} + \underbrace{\frac{k l_0}{2\rho_g}}_{> 0} \Big)}$$

For $k = 0$,

$$\ell^* = \frac{j_+ + j_-}{\lambda S(\bar{f} - \mu_0 + f_0)}$$

bulk depolymerisation



$$\frac{\dot{\gamma}}{\gamma} \propto \frac{\mu_0 - \bar{F}}{F_0} + \alpha$$

contraction