Current fluctuations: mapping non-equilibrium to equilibrium

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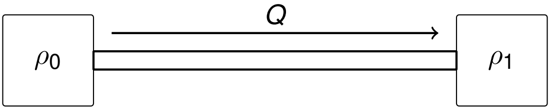
GDR PHENIX – 24th October 2010

Motivations



$$\text{Prob}(\mathcal{C}) \propto \exp\left\{-\frac{\text{energy}(\mathcal{C})}{\text{temperature}}\right\}$$ cannot describe

- Non-equilibrium steady-state

$$\text{Prob}[\rho(x)]$$

- Equilibrium fluctuations of dynamical observables

$$\text{Prob}[Q]$$

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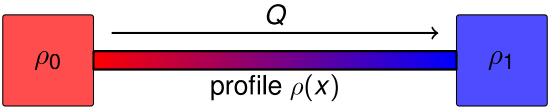

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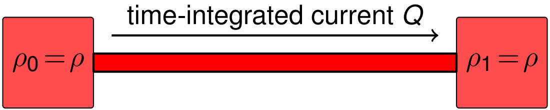

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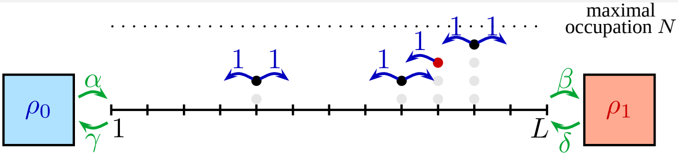
Exclusion processes



- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \le n_i \le N$
- Markov evolution

$$\partial_t P(\{n_i\}) = \sum_{n_i'} \left[W(n_i' \to n_i) P(\{n_i'\}) - W(n_i \to n_i') P(\{n_i\}) \right]$$

- Large deviation function of the time-integrated current Q

$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \qquad (\Leftrightarrow \text{ determining } P(Q))$$

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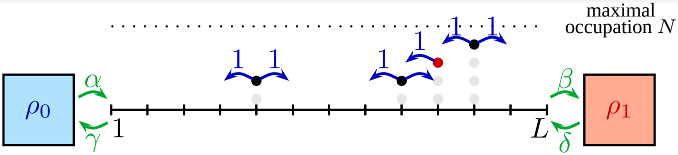

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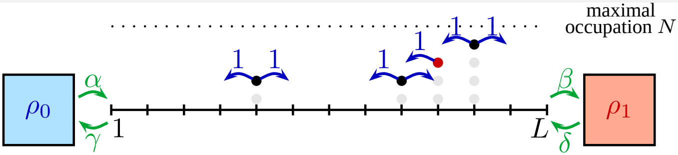

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Example 1: distribution of current in the WASEP

- Periodic boundary condition (mean density ρ)
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- Distribution of the total current: (space&time integrated)

$$\frac{1}{t}\langle Q_{\text{tot}}\rangle = L\,E$$

$$\frac{1}{t}\langle Q_{\text{tot}}^2\rangle_c = L\,2\rho(1-\rho)$$

$$\frac{1}{t}\langle Q_{\text{tot}}^4\rangle_c = L^2\,2[\rho(1-\rho)]^2$$

(free particles would be $\sim L$)

$$\vdots$$

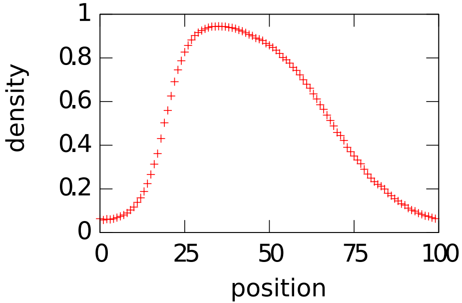
$$\frac{1}{t}\langle Q_{\text{tot}}^k\rangle_c \sim L^{k-2}$$

(**anomalous** scaling in L)

$$\vdots$$

Example 1: distribution of current in the WASEP

- Periodic boundary condition (mean density ρ)
- Bulk field E: small asymmetry ($\sim \frac{1}{L}$) in the jump rates $\rightarrow$non-eq.
- Optimal profile leading to a large atypical flow $J \equiv \frac{1}{t}\langle Q \rangle$:
  - for $|J - E/2| < J_c$: flat profile
  - for $|J - E/2| > J_c$: heterogeneous profile



Non-steady and non-uniform density profile

Example 1: distribution of current in the WASEP

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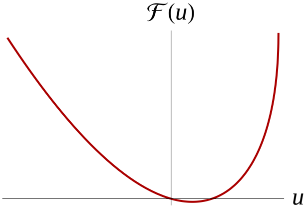
- Bulk field E: small asymmetry ($\sim \frac{1}{L}$) in the jump rates $\rightarrow$non-eq.
- With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122 (2008)]

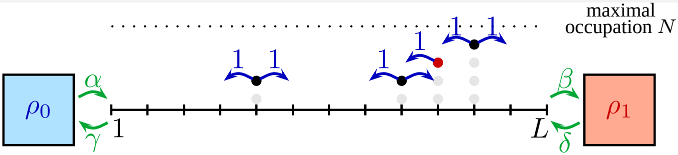
$$\psi(s) = \underbrace{\tfrac{1}{2}s(s-E)\frac{\langle Q^2\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\text{small fluctuations (determinant)}} \quad \text{with } u = \underbrace{-s(s-E)\frac{\sigma\sigma''}{16D^2}}_{\text{can become } > 0}$$



Dynamical phase transition
between
stationnary and non-stationnary
profiles

What about
boundary driven
non-equilibrium transport?

Operator representation [Schütz & Sandow PRE **49** 2726]



$$\partial_t P = \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\leq k\leq L-1} \left[S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k) \right]$$
$$+ \alpha \left[S_1^+ - (1-\hat{n}_1) \right] + \gamma \left[S_1^- - \hat{n}_1 \right]$$
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$S^\pm$ and **creation and annihilation operators**:

$$S^+|n\rangle = (N-n)|n+1\rangle \qquad S^-|n\rangle = n|n-1\rangle$$

Operator representation

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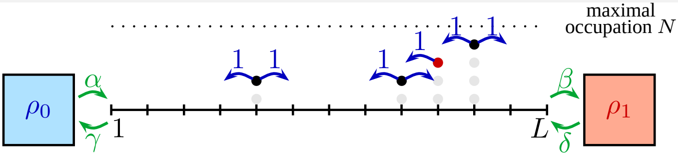

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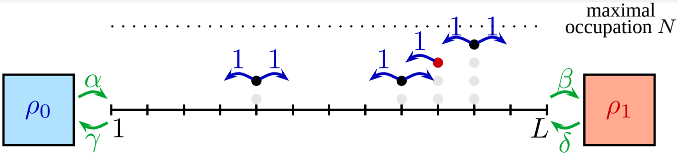
$S^\pm$ and $S^z = \hat{n} - \frac{N}{2}$ are **spin operators** (with $j = \frac{N}{2}$)

Large deviation function



$$\left\langle e^{-sQ} \right\rangle \sim e^{t\psi(s)} \qquad \text{with} \qquad \psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

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Mapping non-eq to eq [Imparato, VL, van Wijland, PTP **184** 276 (2010)]

Large deviations of the current $\psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)$

$$\mathbb{W}(s) = \overbrace{\sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}} + \alpha \left[S_1^+ - (1-\hat{n}_1)\right] + \gamma \left[S_1^- - \hat{n}_1\right] + \delta \left[S_L^+ e^{s} - (1-\hat{n}_L)\right] + \beta \left[S_L^- e^{-s} - \hat{n}_L\right]$$

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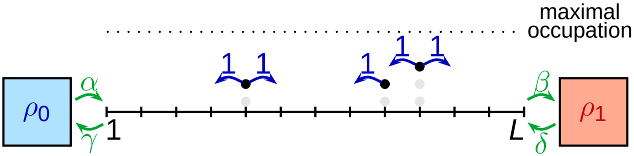
Local transformation

$$\mathcal{Q}^{-1}\mathbb{W}(s)\mathcal{Q} = \sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}$$
$$+ \alpha' \left[S_1^+ - (1-\hat{n}_1)\right] + \gamma' \left[S_1^- - \hat{n}_1\right]$$
$$\underbrace{+\delta' \left[S_L^+ e^{s'} - (1-\hat{n}_L)\right] + \beta' \left[S_L^- e^{-s'} - \hat{n}_L\right]}_{\text{describes contact with reservoirs of same densities}}$$

Summary [Imparato, VL, van Wijland, PRE **80** 011131 & PTP **184** 276 (2010)]

Symmetric exclusion process

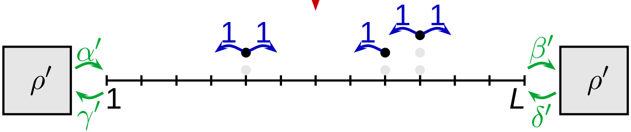


Local transformation

$\mathrm{Prob}^{\text{stationn.}}_{\text{non-eq}}(\text{current})$

$\updownarrow$

$\mathrm{Prob}^{\text{stationn.}}_{\text{eq}}(\text{current}')$



System in equilibrium

Result

[Imparato, VL, van Wijland, PRE **80** 011131 & PTP **184** 276 (2010)]

- Large deviation function

[Reminder: $\langle e^{-sQ} \rangle \sim e^{t\psi(s)}$]

$$\psi(s) = \underbrace{\frac{1}{L}\mu(s)}_{\text{saddle}} + \underbrace{\frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(s)\right)}_{\text{fluctuations}}$$

$$\mu(s) = -\big(\text{arcsinh}\,\sqrt{\omega}\big)^2 \qquad \omega = (1-e^{s})\big(e^{-s}\rho_0-\rho_1-(e^{-s}-1)\rho_0\rho_1\big)$$

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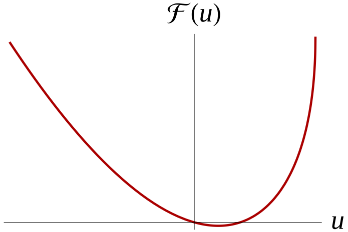
- Universal function $\mathcal{F}$

$$\mathcal{F}(u) = \sum_{k\geq 2}\frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$



Summary

Approach:

- large deviation function
- operator formalism
- fluctuating hydrodynamics

Results:

- universal fluctuations
- possible dynamical phase transition
- non-eq. to eq. mapping

Open questions:

$\rightarrow$ Eq$\leftrightarrow$non-eq mapping in higher dimensions?
$\rightarrow$ More generic systems of interacting particle?
$\rightarrow$ Link to other eq$\leftrightarrow$non-eq mappings?
$\rightarrow$ Crossover to KPZ? Universal fluctuations?

Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

For **periodic boundary conditions**

- large deviation function

$$\psi(s) = \underbrace{L^{-1}\rho(1-\rho)s^2}_{\text{order 0}} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} \quad \text{with} \quad u = -\tfrac{1}{2}\rho(1-\rho)s^2$$

- universal function

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

- scaling of cumulants of the total current Q_{tot}

$$\tfrac{1}{t}\langle Q_{\text{tot}}^2\rangle \sim L$$

$$\tfrac{1}{t}\langle Q_{\text{tot}}^{2k}\rangle \sim L^{2k-2} \qquad (k \geq 2)$$

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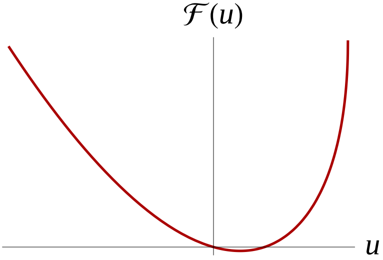

With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
Driving field E

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Periodic boundary conditions
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For the WASEP:
$D = 1$, $\sigma = 2\rho(1-\rho)$

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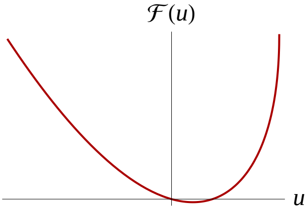

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[Imparato, VL, van Wijland, **PRE** 80 011131]

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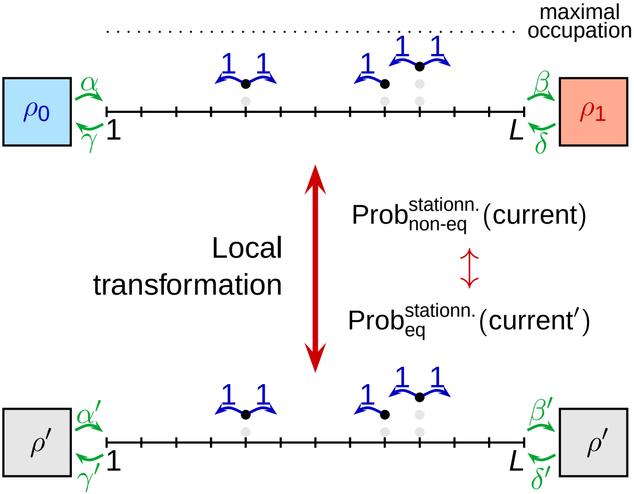
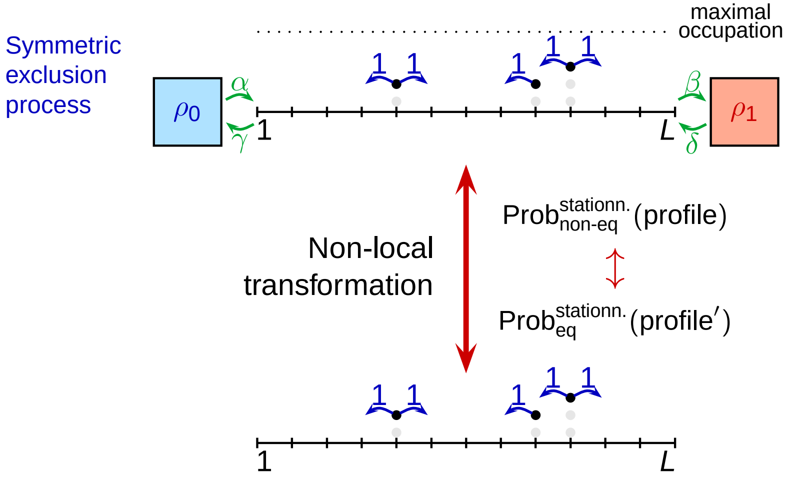


System in equilibrium

For the density profile

[Tailleur, Kurchan, VL, JPA **41** 505001]



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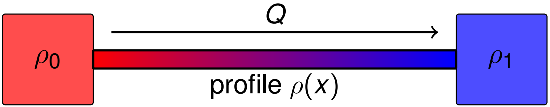
Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

- long-range correlations
- breaking of time-reversal symmetry

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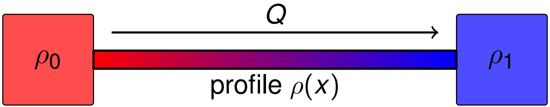

Boundary-driven transport model:

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Non-local mapping to equilibrium:

- accounts for long-range correlations
- (density gradient)$_{\text{non-eq.}}$ $\longleftrightarrow$ (fixed density)$_{\text{eq.}}$
- yields Prob[$\rho(x)$] through a extremalization principle