Homework 1 - Elements of correction

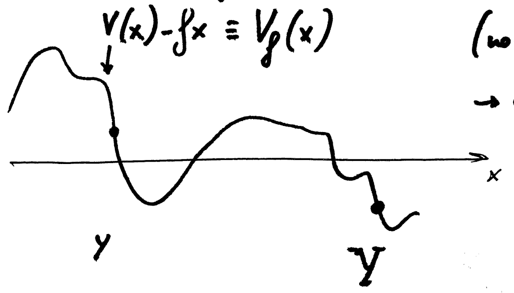
Exercice 1 - correction

1. Particle in a tilted potential $\quad x \in [0,1]$ periodic b.c.

$$\partial_t x = -V'(x) + f + \eta(t) \qquad F(x) = -V'(x) + f \qquad \langle \eta(t)\eta(t') \rangle = 2T\,\delta(t-t')$$

st.st. : $-\partial_x \left(F(x) P_{st}(x) \right) + T\,\partial_x^2 P_{st}(x) = 0$

Instead of solving $P_{st}(x)$ as in Le Doussal & Vinokur or as in Scheidl, one may follow a 1st passage time approach (mean — : MFPT)



(note that $V_f(x)$ is not a continuous periodic function)
→ one transforms the problem into a problem with $x \in \mathbb{R}$

| y = starting point
| Y = passage point

$t_1(y \rightsquigarrow Y)$ = mean 1st passage time in Y, starting from y at time 0

One has $\underbrace{\langle t_1(y \rightsquigarrow Y) \rangle}_{\equiv \tau_1(y)} = \int_0^{+\infty} dt\; t\; \mathrm{Prob}\left(t_1(y \rightsquigarrow Y) = t\right)$ → This is a cumulative distribut° funct°

$$= \int_0^{+\infty} dt\; t \cdot \left(-\partial_t \mathrm{Prob}\left(t_1(y \rightsquigarrow Y) \geq t \right) \right) = 0 + \int_0^{+\infty} dt\; \mathrm{Prob}\left(t_1(y \rightsquigarrow Y) \geq t \right)$$

$\underbrace{\mathrm{Prob}\left(t_1(y \rightsquigarrow Y) \geq t\right)}_{\equiv Q(y,t)} = \int_{-\infty}^{Y} dy'\, \underbrace{P_{abs}(y', t | y, 0)}$ ⇐ $\begin{cases} = \text{probability that the particle goes in } Y \text{ for the 1st time, having started in } y \text{, before } t \text{ (for the process absorbing in } Y) \\ = \text{probability that the particle is still in any } y' \in]-\infty, Y[\end{cases}$

condit. prob. density for the process where the particle is absorbed in Y

It verifies the Backward Fokker-Planck equation

$$\boxed{\partial_t P_{abs}(y', t | y, 0) = F(y)\,\partial_y P_{abs}(y', t | y, 0) + T\,\partial_y^2 P_{abs}(y', t | y, 0)}$$

Hence $\quad \partial_t Q(y,t) = F(y)\,\partial_y Q(y,t) + T\,\partial_y^2 Q(y,t) \quad$ with $Q(Y,t) = 0$

$\int_0^{+\infty} dt \curvearrowright \quad \underbrace{Q(y, +\infty)}_{=0} - \underbrace{Q(y, 0)}_{=1} = F(y)\,\partial_y \tau_1(y) + T\,\partial_y^2 \tau_1(y)$

Finally, the equation on the MFPT is : $\boxed{F(y)\,\partial_y \tau_1(y) + T\,\partial_y^2 \tau_1(y) = -1, \quad \tau_1(Y) = \tau_1(-\infty) = 0}$

(Pontryagin equation)

One finds $\quad \boxed{\tau_1(y) = \frac{1}{T} \int_y^Y dy' \int_{-\infty}^{y'} dy''\, \frac{Z^f(y'')}{Z^f(y')}}$

$Z^f(y_0) = e^{-\frac{1}{T} V_f(y_0)} = e^{-\frac{1}{T}(V(y_0) - f y_0)}$

= Boltzman weight if one were in equilibrium

1- Use of the formula for the MFPT: $\rho = 1$ to simplify

Exercise 1 - correction (2

in our case $$\frac{Z_f(y'')}{Z_f(y')} = \underbrace{e^{\beta f (y''-y')}}_{\text{this part is not periodic}} \cdot \underbrace{e^{-\beta [V(y'')-V(y')]}}_{\text{this part is periodic of period 1}}$$

one could of course take any period.

in the expression of $\tau_1(y)$ we thus set $y'' = y' - y_0$, $y_0 \in [0,+\infty[$ so that, inverting integrations

$$\tau_1(0) = \beta \int_0^{+\infty} dy_0\, e^{-\beta f y_0} \underbrace{\int_0^{Y} dy'\, \underbrace{e^{-\beta[V(y'-y_0)-V(y')]}}_{\text{periodic: invariant by translation } y' \to y'+1}}_{\text{for } Y\in\mathbb{N} \text{ this integral is thus } Y \times \int_0^1 dy'\, e^{-\beta[V(y'-y_0)-V(y')]}}$$

We thus obtain that for $Y \in \mathbb{N}$, the MFPT from 0 to Y is proportional to Y. As Y represents the distance from 0 to Y, this allows to identify the mean velocity $\bar{v}$ as $\bar{v} = \frac{Y}{\tau_1(0 \to Y)} = \frac{1}{\tau_1(0\to 1)}$. In other words,

$$\boxed{\bar{v}^{-1} = \beta \int_0^{+\infty} dy_0\, e^{-\beta f y_0} \int_0^1 dy'\, e^{-\beta[V(y'-y_0)-V(y')]}}$$

This expression is generic for any 1-periodic potential $V(x)$.
Computations can be pushed forward for the cosine potential:

$$V(y'-y_0) - V(y') = 2\sin(\pi y_0)\sin[\pi(y_0 - 2y')]$$

Using now that for any periodic function $\varphi(y)$ of period 1, $\int_0^1 dy'\varphi(y'+a) = \int_a^{1+a} ds'\varphi(s') = \int_0^1 ds'\varphi(s')$, one can translate $y' \to y' - y_0/2$ in the integral and this yields

$$\bar{v}^{-1} = \beta \int_0^{+\infty} dy_0\, e^{-\beta f y_0} \underbrace{\int_0^1 dy'\, e^{2\beta \sin(\pi y_0)\sin(2\pi y')}}_{I_0(2\beta \sin \pi y_0)}$$

$I_0(x)$ = Bessel function I

$$\boxed{\bar{v}^{-1} = \beta \int_0^{+\infty} dy_0\, e^{-\beta f y_0}\, I_0(2\beta \sin \pi y_0)}$$

See the mathematica file for some plots