```
(*mean velocity for the tilted particle :
  analytical results from the Mean First Passage Time (MFPT) approach, see correction*)
$Assumptions = p > 0 && p ∈ Integers && y0 > 0
```

```
p > 0 && p ∈ Integers && y0 > 0
```

```
V[y_] := Cos[2 π p y]
```

```
Integrate[Exp[-β (V[yp - y0] - V[yp])], {yp, 0, 1}]
```

$$\int_0^1 e^{-\beta\,(-\mathrm{Cos}[2\,p\,\pi\,yp]+\mathrm{Cos}[2\,p\,\pi\,(-y0+yp)])}\,dyp$$

```
TrigFactor[(V[yp - y0] - V[yp]) /. p → 1]
```

```
2 Sin[π yp - π (-y0 + yp)] Sin[π yp + π (-y0 + yp)]
```

```
ΔV = Simplify /@ %
```

```
-2 Sin[π y0] Sin[π (y0 - 2 yp)]
```

```
ΔV = -2 Sin[π y0] Sin[π (0 y0 - 2 yp)]
```

```
2 Sin[π y0] Sin[2 π yp]
```

```
Integrate[Exp[-β ΔV], {yp, 0, 1}]
```

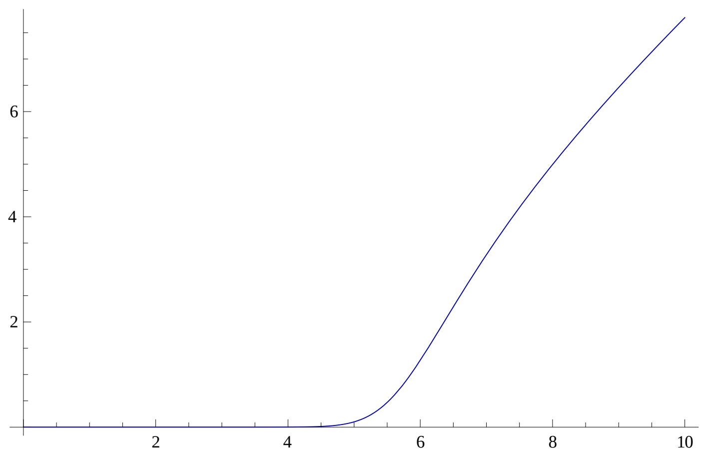
```
BesselI[0, 2 β Sin[π y0]]
```

```
vbar[β_, f_] := 1 / (β NIntegrate[Exp[-β f y0] BesselI[0, 2 β Sin[π y0]],
      {y0, 0, ∞}, MaxRecursion → 10, WorkingPrecision → 10])
```

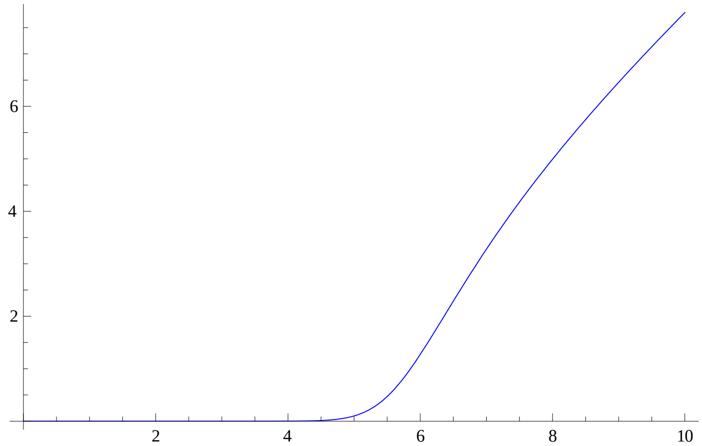
```
βnum = 5; vbar[βnum, 8]
```

```
5.292430574
```

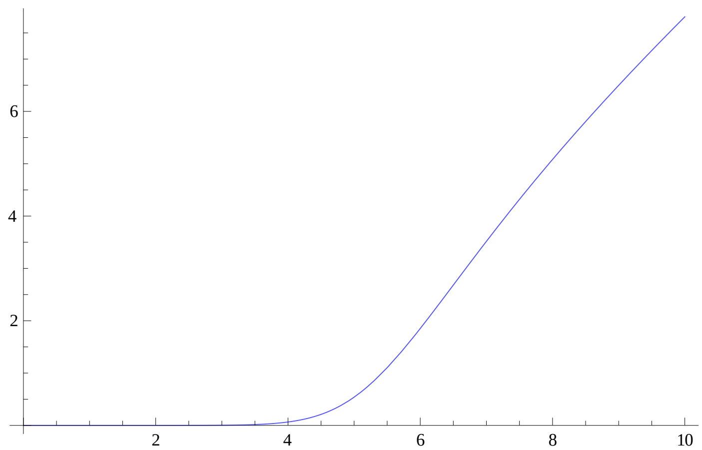
```
βnum = 40;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Darker@Blue]
```



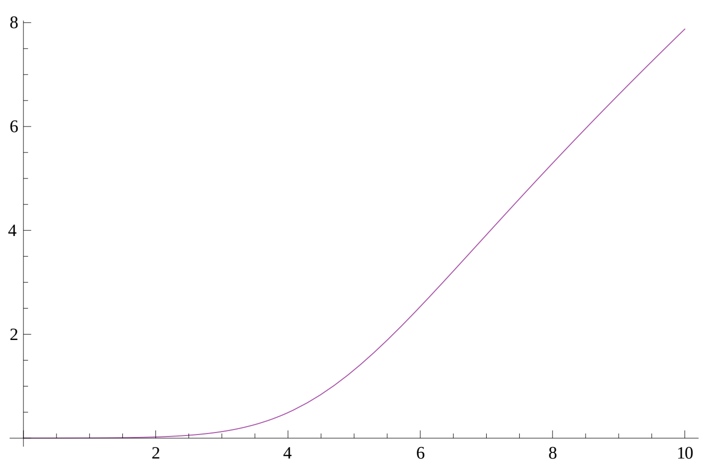
```
βnum = 20;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Blue]
```



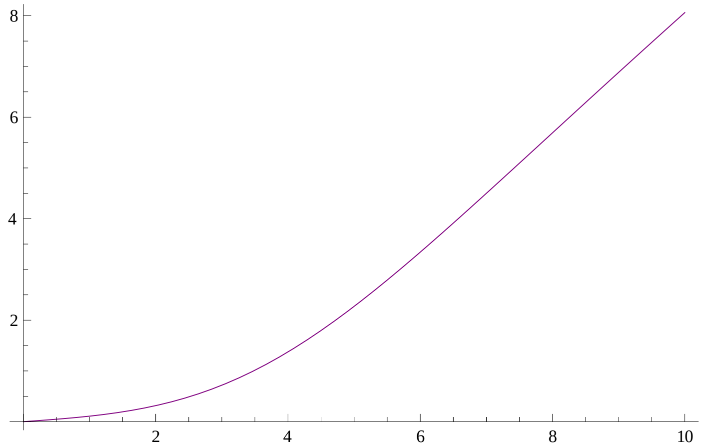

```
βnum = 10;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Lighter@Blue]
```



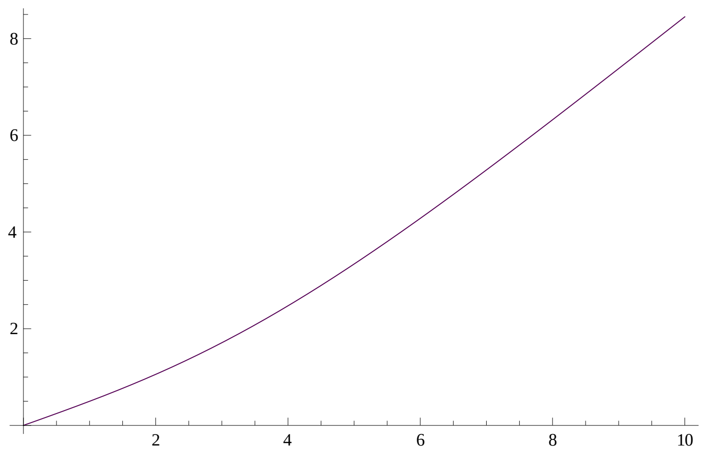

```
βnum = 5;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Lighter@Purple]
```



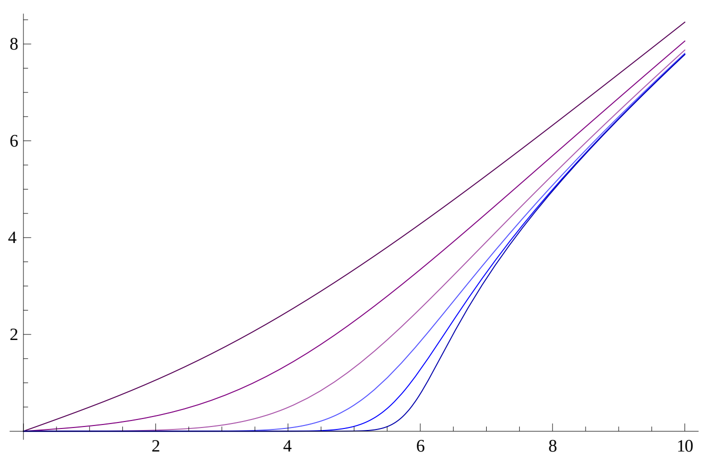
```
βnum = 5 / 2;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Purple]
```




```
βnum = 5 / 4;
plot[βnum] = Plot[Quiet@vbar[βnum, f], {f, 0, 10}, PlotStyle → Darker@Purple]
```




```
Show[plot /@ {5 / 4, 5 / 2, 5, 10, 20, 40}]
```

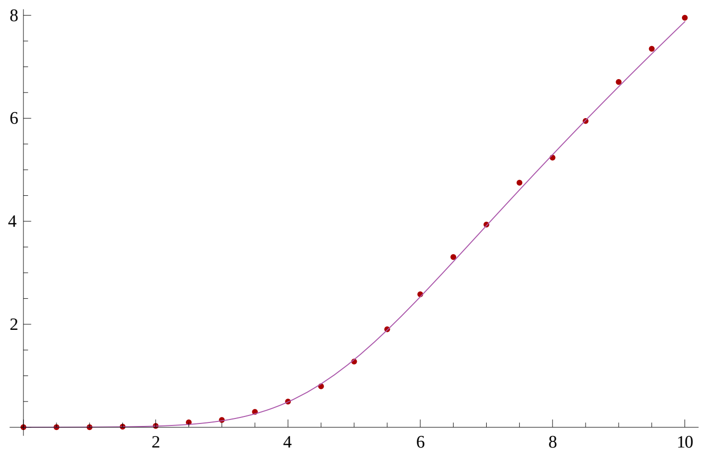



```
(*numerical simulations of the Langevin equation in discrete time and continuous space*)
```

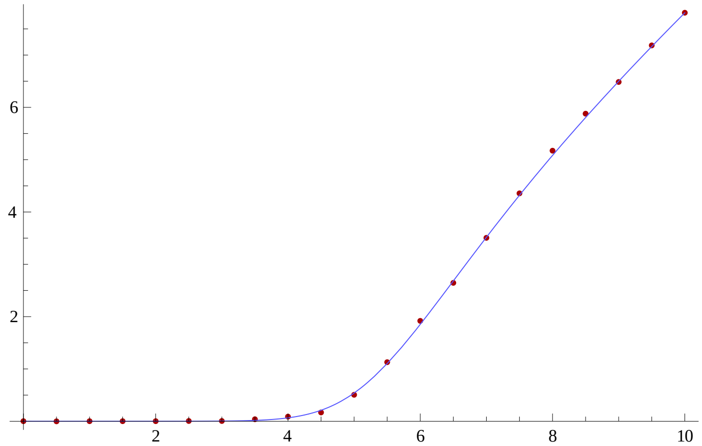
```
V[y_] = Cos[2 π y]

Cos[2 π y]

F[x_, f_] = -V'[x] + f

f + 2 π Sin[2 π x]

βnum = 40; Tnum = 1 / βnum;

δtnum = 1 / 100;
steps = 3 × 10^4;
tmax = δtnum steps;
σnum = Sqrt[2 Tnum δtnum];

voff[f_] := Module[{fnum, tabeta0},
  tabeta0 = RandomVariate[NormalDistribution[0, σnum], steps + 1];
  fnum = f;
  xnum = N[x0num, 10]; index = 1;
  For[tnum = 0, tnum ≤ tmax, tnum += δtnum; index ++,
   eta0 = tabeta0[[index]];
   xnum = xnum + δtnum F[xnum, fnum] + eta0;
  ];
  xnum / tmax]

Timing[voff[8]]

{0.336021, 5.00537}

Timing[tv[βnum] = Table[{f, voff[f]}, {f, 0, 10, 1/2}]]
```

$\{6.70842, \{\{0, -0.00169558\}, \{\frac{1}{2}, 0.0015004\}, \{1, 0.00167077\}, \{\frac{3}{2}, 0.00186926\}, \{2, 0.00164114\}, \{\frac{5}{2}, 0.0017472\}, \{3, 0.00211456\}, \{\frac{7}{2}, 0.0020656\}, \{4, 0.00196859\}, \{\frac{9}{2}, 0.00186954\}, \{5, 0.0023855\}, \{\frac{11}{2}, 0.0922913\}, \{6, 0.779069\}, \{\frac{13}{2}, 2.07403\}, \{7, 3.1861\}, \{\frac{15}{2}, 4.11567\}, \{8, 4.96094\}, \{\frac{17}{2}, 5.78882\}, \{9, 6.44915\}, \{\frac{19}{2}, 7.1263\}, \{10, 7.83724\}\}\}$

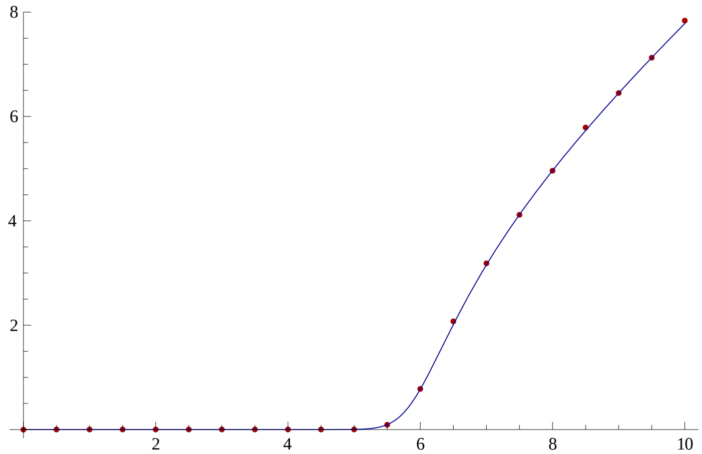
```
Show[ListPlot[tv[5], PlotStyle → Darker@Red], plot[5]]
```



```
Show[ListPlot[tv[10], PlotStyle→Darker@Red], plot[10]]
```




```
Show[ListPlot[tv[40], PlotStyle→Darker@Red], plot[40]]
```




```
(*From discrete master equation to Fokker Planck*)
V[x_] = Cos[2 p π x]
```

Cos[2 p π x]

```
pplus[k_] = δt / δx^2 T Exp[-1 / (2 T) (V[(k + 1) / L] - V[k / L] - f / L)]
```

$$\frac{e^{-\frac{-\frac{f}{L}-\text{Cos}\left[\frac{2 k p \pi}{L}\right]+\text{Cos}\left[\frac{2 (1+k) p \pi}{L}\right]}{2 T}} T \delta t}{\delta x^2}$$

```
pmoins[k_] = δt / δx^2 T Exp[-1 / (2 T) (V[(k - 1) / L] - V[k / L] + f / L)]
```

$$\frac{e^{-\frac{\frac{f}{L}+\text{Cos}\left[\frac{2 (-1+k) p \pi}{L}\right]-\text{Cos}\left[\frac{2 k p \pi}{L}\right]}{2 T}} T \delta t}{\delta x^2}$$

```
P0[k_] = P[k / L]
```

$$P\left[\frac{k}{L}\right]$$

```
(*right hand side of the master equation*)
RHS = pplus[k - 1] P0[k - 1] + pmoins[k + 1] P0[k + 1] - (pplus[k] + pmoins[k]) P0[k]
```

$$\frac{e^{-\frac{-\frac{f}{L}-\mathrm{Cos}\left[\frac{2(-1+k)p\pi}{L}\right]+\mathrm{Cos}\left[\frac{2kp\pi}{L}\right]}{2T}}\,T\,\delta t\,P\left[\frac{-1+k}{L}\right]}{\delta x^2}-\left(\frac{e^{-\frac{\frac{f}{L}+\mathrm{Cos}\left[\frac{2(-1+k)p\pi}{L}\right]-\mathrm{Cos}\left[\frac{2kp\pi}{L}\right]}{2T}}\,T\,\delta t}{\delta x^2}+\frac{e^{-\frac{-\frac{f}{L}-\mathrm{Cos}\left[\frac{2kp\pi}{L}\right]+\mathrm{Cos}\left[\frac{2(1+k)p\pi}{L}\right]}{2T}}\,T\,\delta t}{\delta x^2}\right)P\left[\frac{k}{L}\right]+\frac{e^{-\frac{\frac{f}{L}+\mathrm{Cos}\left[\frac{2kp\pi}{L}\right]-\mathrm{Cos}\left[\frac{2(1+k)p\pi}{L}\right]}{2T}}\,T\,\delta t\,P\left[\frac{1+k}{L}\right]}{\delta x^2}$$

```
(*preparation of the large L expansion at fixed x = k/L*)
RHSx = RHS /. k → x L
```

$$-\left(\frac{e^{-\frac{-\frac{f}{L}-\mathrm{Cos}[2p\pi x]+\mathrm{Cos}\left[\frac{2p\pi(-1+Lx)}{L}\right]}{2T}}\,T\,\delta t}{\delta x^2}+\frac{e^{-\frac{-\frac{f}{L}-\mathrm{Cos}[2p\pi x]+\mathrm{Cos}\left[\frac{2p\pi(1+Lx)}{L}\right]}{2T}}\,T\,\delta t}{\delta x^2}\right)P[x]+\frac{e^{-\frac{\frac{f}{L}+\mathrm{Cos}[2p\pi x]-\mathrm{Cos}\left[\frac{2p\pi(-1+Lx)}{L}\right]}{2T}}\,T\,\delta t\,P\left[\frac{-1+Lx}{L}\right]}{\delta x^2}+\frac{e^{-\frac{\frac{f}{L}+\mathrm{Cos}[2p\pi x]-\mathrm{Cos}\left[\frac{2p\pi(1+Lx)}{L}\right]}{2T}}\,T\,\delta t\,P\left[\frac{1+Lx}{L}\right]}{\delta x^2}$$

```
Simplify[Series[RHSx, {L, ∞, 2}]]
```

$$-\frac{1}{\delta x^2 L^2}\delta t\left(4p^2\pi^2\,\mathrm{Cos}[2p\pi x]\,P[x]+\left(f+2p\pi\,\mathrm{Sin}[2p\pi x]\right)P'[x]-T\,P''[x]\right)+O\left[\frac{1}{L}\right]^3$$

```
Simplify[Normal@% / δt /. δx → 1 / L]
```

$$-4p^2\pi^2\,\mathrm{Cos}[2p\pi x]\,P[x]-\left(f+2p\pi\,\mathrm{Sin}[2p\pi x]\right)P'[x]+T\,P''[x]$$

```
(*one reckognizes*)
D[(-f + V'[x]) P[x], x]
```

$$-4p^2\pi^2\,\mathrm{Cos}[2p\pi x]\,P[x]+\left(-f-2p\pi\,\mathrm{Sin}[2p\pi x]\right)P'[x]$$

```
(*numerical simulations of the Langevin equation in discrete time and discrete space*)

(*p=1*)
V[y_] = Cos[2 π y]
```

$\mathrm{Cos}[2\pi y]$

```
F[x_, f_] = -V'[x] + f
```

$f+2\pi\,\mathrm{Sin}[2\pi x]$

```
pplusx[x_] = δt / δx^2 T Exp[-1 / (2 T) (V[x + 1 / L] - V[x] - f / L)]
```

$$\frac{e^{-\frac{-\frac{f}{L}-\mathrm{Cos}[2\pi x]+\mathrm{Cos}\left[2\pi\left(\frac{1}{L}+x\right)\right]}{2T}}\,T\,\delta t}{\delta x^2}$$

```
pmoinsx[x_] = δt / δx^2 T Exp[-1 / (2 T) (V[x - 1 / L] - V[x] + f / L)]
```

$$\frac{e^{-\frac{\frac{f}{L}-\mathrm{Cos}[2\pi x]+\mathrm{Cos}\left[2\pi\left(-\frac{1}{L}+x\right)\right]}{2T}}\,T\,\delta t}{\delta x^2}$$

```
βnum = 20; Tnum = 1 / βnum;
```

```
δtnum = 10^-4;
steps = 10^5;
tmax = δtnum steps;
Lnum = 100; δxnum = 1 / Lnum;
```

```
(*check of the worst rate to jump to the right ;
we lineraize |V((k+1)/L)-V(k/L) | ~1/L and takef < 20 ;
the following probability has to be  <1/2 *)
```

```
N[δtnum / δxnum^2 Tnum Exp[-1 / (2 Tnum) (-1 / Lnum - 10 / Lnum)]]
```

0.150208

```
(*check of the worst rate to jump to the right ;
we lineraize |V((k+1)/L)-V(k/L) | ~1/L and takef < 20 ;
the following probability has to be  <1/2 *)
```

```
N[δtnum / δxnum^2 Tnum Exp[-1 / (2 Tnum) (-1 / Lnum + 0 / Lnum)]]
```

0.0203285

```
voffdiscrete[ff_] := Module[{fnum, tabeta0, Pplus, Pmoins, jump},
  fnum = ff;
  xnum = N[0, 10];
  For[tnum = 0, tnum ≤ tmax, tnum += δtnum,
   Pplus = pplusx[xnum] /. {T → Tnum, δt → δtnum, δx → δxnum, f → fnum, L → Lnum};
   Pmoins = pmoinsx[xnum] /. {T → Tnum, δt → δtnum, δx → δxnum, f → fnum, L → Lnum};
   (*Print[N/@{Pplus,Pmoins}];*)
   jump = RandomChoice[{Pplus, Pmoins, 1 - Pmoins - Pplus} -> {δxnum, -δxnum, 0}];
   xnum = xnum + jump;
  ];
  xnum / tmax]
```

```
N@Timing[voffdiscrete[8]]
```

{18.1451, 5.368}

```
vbar[βnum, 8]
```

4.993392683

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

{200.237, {{0., -0.048}, {1., -0.043}, {2., 0.061}, {3., 0.054}, {4., 0.06},
 {5., 0.163}, {6., 1.168}, {7., 3.587}, {8., 5.456}, {9., 6.886}, {10., 8.805}}}

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

{203.329, {{0., -0.054}, {1., 0.047}, {2., 0.056}, {3., 0.059}, {4., 0.063},
 {5., 0.065}, {6., 1.471}, {7., 3.775}, {8., 5.069}, {9., 7.084}, {10., 8.557}}}

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

{202.549, {{0., -0.057}, {1., -0.046}, {2., 0.056}, {3., 0.059}, {4., 0.068},
 {5., 0.167}, {6., 1.063}, {7., 3.382}, {8., 5.248}, {9., 7.178}, {10., 8.861}}}

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

{199.468, {{0., -0.048}, {1., 0.057}, {2., 0.056}, {3., 0.055}, {4., 0.064},
 {5., 0.079}, {6., 1.868}, {7., 3.69}, {8., 4.97}, {9., 7.112}, {10., 8.73}}}

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

{202.381, {{0., 0.05}, {1., 0.05}, {2., 0.052}, {3., 0.059}, {4., 0.065},
 {5., 0.17}, {6., 1.567}, {7., 3.477}, {8., 5.02}, {9., 6.689}, {10., 8.502}}}

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

```
{203.849, {{0., -0.052}, {1., 0.053}, {2., 0.054}, {3., 0.061}, {4., 0.057},
  {5., 0.165}, {6., 1.675}, {7., 3.672}, {8., 5.685}, {9., 7.19}, {10., 8.863}}}
```

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

```
{203.349, {{0., -0.054}, {1., 0.058}, {2., -0.049}, {3., 0.057}, {4., 0.06},
  {5., 0.064}, {6., 0.967}, {7., 3.483}, {8., 5.449}, {9., 7.476}, {10., 8.86}}}
```

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

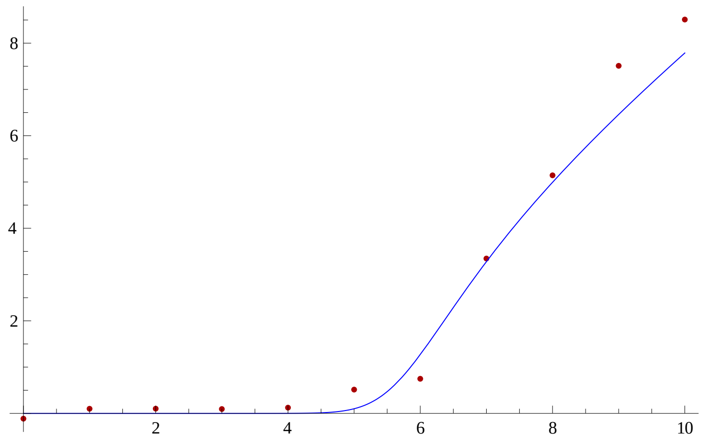
```
{199.984, {{0., 0.049}, {1., 0.045}, {2., 0.052}, {3., 0.055}, {4., 0.064},
  {5., 0.169}, {6., 0.875}, {7., 3.473}, {8., 5.372}, {9., 6.986}, {10., 8.665}}}
```

```
Timing[tvdiscrete[βnum] = N@Table[{f, voffdiscrete[f]}, {f, 0, 10, 1}]]
```

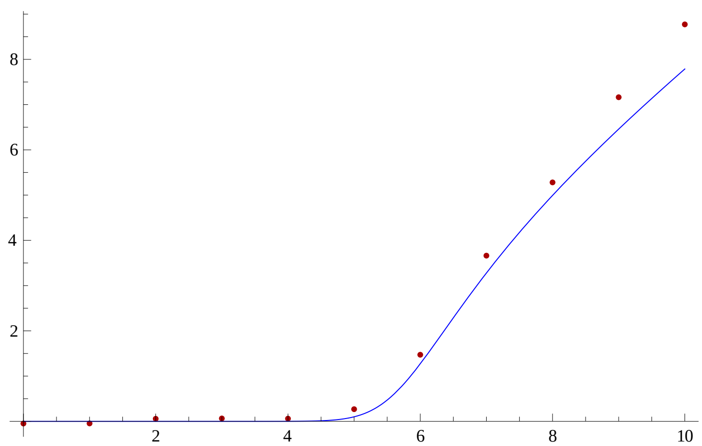
```
{199.516, {{0., -0.048}, {1., 0.049}, {2., 0.056}, {3., 0.065}, {4., 0.066},
  {5., 0.271}, {6., 1.065}, {7., 3.064}, {8., 5.48}, {9., 7.27}, {10., 8.941}}}
```

```
run[1] = {{0.`, -0.114`}, {1.`, 0.1`}, {2.`, 0.102`}, {3.`, 0.094`}, {4.`, 0.124`},
   {5.`, 0.514`}, {6.`, 0.748`}, {7.`, 3.346`}, {8.`, 5.144`}, {9.`, 7.51`}, {10.`, 8.51`}};
```

```
Show[ListPlot[tvdiscrete[20], PlotStyle→Darker@Red, PlotRange→All], plot[20]]
```

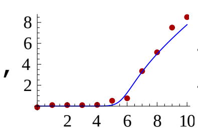



```
Show[ListPlot[tvdiscrete[20], PlotStyle→Darker@Red, PlotRange→All], plot[20]]
```




```
(*which is not too bad*)
```

```
(*averaging now over 10 runs*)

run[1] = {{0.`, -0.114`}, {1.`, 0.1`}, {2.`, 0.102`}, {3.`, 0.094`}, {4.`, 0.124`},
   {5.`, 0.514`}, {6.`, 0.748`}, {7.`, 3.346`}, {8.`, 5.144`}, {9.`, 7.51`}, {10.`, 8.51`}};

run[2] = {200.237, {{0., -0.048}, {1., -0.043}, {2., 0.061}, {3., 0.054}, {4., 0.06},
     {5., 0.163}, {6., 1.168}, {7., 3.587}, {8., 5.456}, {9., 6.886}, {10., 8.805}}}[[2]];

run[3] = {203.329, {{0., -0.054}, {1., 0.047}, {2., 0.056}, {3., 0.059}, {4., 0.063},
     {5., 0.065}, {6., 1.471}, {7., 3.775}, {8., 5.069}, {9., 7.084}, {10., 8.557}}}[[2]];

run[4] = {202.549, {{0., -0.057}, {1., -0.046}, {2., 0.056}, {3., 0.059}, {4., 0.068},
    {5., 0.167}, {6., 1.063}, {7., 3.382}, {8., 5.248}, {9., 7.178}, {10., 8.861}}}[[2]]

{{0., -0.057}, {1., -0.046}, {2., 0.056}, {3., 0.059}, {4., 0.068},
 {5., 0.167}, {6., 1.063}, {7., 3.382}, {8., 5.248}, {9., 7.178}, {10., 8.861}}

run[5] = {199.468, {{0., -0.048}, {1., 0.057}, {2., 0.056}, {3., 0.055}, {4., 0.064},
    {5., 0.079}, {6., 1.868}, {7., 3.69}, {8., 4.97}, {9., 7.112}, {10., 8.73}}}[[2]]

{{0., -0.048}, {1., 0.057}, {2., 0.056}, {3., 0.055}, {4., 0.064},
 {5., 0.079}, {6., 1.868}, {7., 3.69}, {8., 4.97}, {9., 7.112}, {10., 8.73}}

run[6] = {202.381, {{0., 0.05}, {1., 0.05}, {2., 0.052}, {3., 0.059}, {4., 0.065},
    {5., 0.17}, {6., 1.567}, {7., 3.477}, {8., 5.02}, {9., 6.689}, {10., 8.502}}}[[2]]

{{0., 0.05}, {1., 0.05}, {2., 0.052}, {3., 0.059}, {4., 0.065},
 {5., 0.17}, {6., 1.567}, {7., 3.477}, {8., 5.02}, {9., 6.689}, {10., 8.502}}

run[7] = {203.849, {{0., -0.052}, {1., 0.053}, {2., 0.054}, {3., 0.061}, {4., 0.057},
     {5., 0.165}, {6., 1.675}, {7., 3.672}, {8., 5.685}, {9., 7.19}, {10., 8.863}}}[[2]];

run[8] = {203.349, {{0., -0.054}, {1., 0.058}, {2., -0.049}, {3., 0.057}, {4., 0.06},
     {5., 0.064}, {6., 0.967}, {7., 3.483}, {8., 5.449}, {9., 7.476}, {10., 8.86}}}[[2]];

run[9] = {199.984, {{0., 0.049}, {1., 0.045}, {2., 0.052}, {3., 0.055}, {4., 0.064},
     {5., 0.169}, {6., 0.875}, {7., 3.473}, {8., 5.372}, {9., 6.986}, {10., 8.665}}}[[2]];

run[10] = {199.516, {{0., -0.048}, {1., 0.049}, {2., 0.056}, {3., 0.065}, {4., 0.066},
     {5., 0.271}, {6., 1.065}, {7., 3.064}, {8., 5.48}, {9., 7.27}, {10., 8.941}}}[[2]];

imax = 10;

Show[ListPlot[Mean[run /@ Array[(#1) &, imax]], PlotStyle → Darker@Green],
```




```
]
```