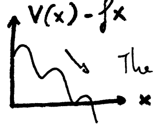
• Exercice for october 16th : toy model for the depinning transition

1- Problem in continuous space : a particle in a tilted periodic potential

$x \in [0,1]$ with periodic boundary conditions

periodic potential $V(x) = \cos(2\pi p x)$ p integer

constant uniform "drift" force f



The particle slides down

$$\boxed{\partial_t x = -V'(x) + f + \eta(t)}$$

$F(x) = -V'(x) + f$

$\langle \eta(t)\eta(t')\rangle = 2T\,\delta(t-t')$

Corresponding Fokker-Planck equation for the steady state $P_{st}(x)$:

$$0 \; \cancel{\partial_t P(x,t)}^{\,0} = \boxed{-\partial_x\left(F(x)P_{st}(x)\right) + T\partial_x^2 P_{st}(x) = 0}$$

⚠ Note that a naive Boltzmann solution $P_{eq}(x) = e^{-\frac{1}{T}(V(x)-fx)}$ does not work because it does not respect boundary conditions

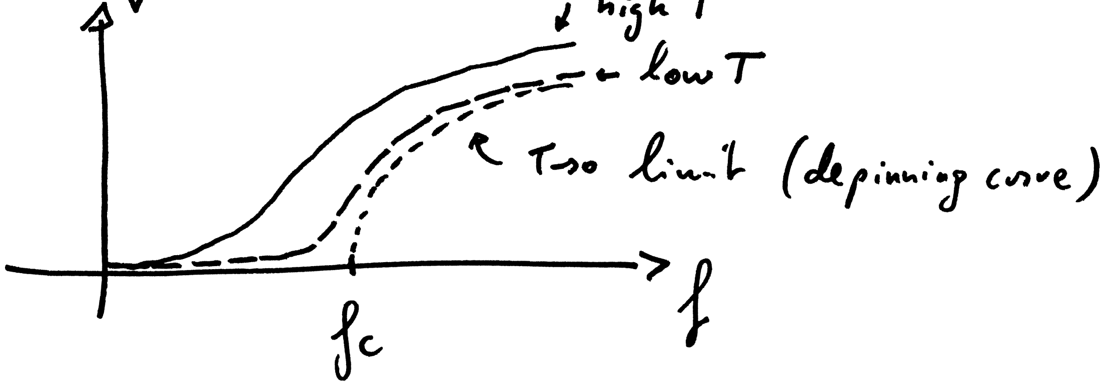
→ The problem is a real 1D non-equilibrium process.

Q1a: Find the steady-state $P_{st}(x)$

Hint (i) : use that the probability current $F(x)P_{st}(x) + T\partial_x P_{st}$ is a uniform constant j (as seen from the steady state equation)

Hint (ii) : read
- P. Le Doussal & V. Vinokur, Physica C (1995) 254 63
- or S. Scheidl, Z. Phys. B (1995) 97 345

Q1b : Determine the mean velocity $\bar{v} = \langle \partial_t x \rangle_{st}$ as a function of f & T



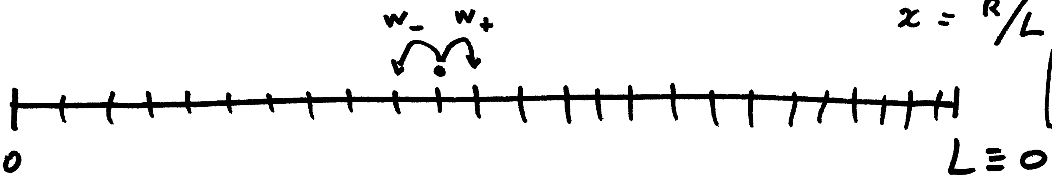
The $T\to 0$ limit describes a 1D non-equilibrium $\boxed{\text{phase transition}}$

Exercice 1

. 2- Numerical simulations in discrete time & space :

Time step δt

lattice with a large number of sites $0 \leq k \leq L$.



$x = k/L \in [0,1]$

$$\boxed{\delta x = 1/L}$$

Periodic boundary conditions: $k=L$ identified to $k=0$.

Measure the mean-velocity through the total current $\mathring{Q}$

$$\mathring{Q} = (\#\text{ jumps to the right}) - (\#\text{ jumps to the left}) \to \begin{cases} Q \mapsto Q+1 & \text{for } \curvearrowright \\ Q \mapsto Q-1 & \text{for } \curvearrowleft \end{cases}$$

→ $\bar{v}$ is recovered from $\bar{v} = \frac{1}{t} \langle \mathring{Q} \rangle \delta x$ (and take large t, average over many runs)

(The difference b/w k and $\mathring{Q}$ is that k is computed modulo L, while $\mathring{Q}$ is not)

. Transition ~~rates~~ probabilities: take (not obvious to find, since we are out of equilibrium)

$$\boxed{\begin{aligned} \text{prob}(k \to k+1) &\equiv p_+(k) = \frac{\delta t}{\delta x^2} T e^{-\frac{1}{2T}\left(V\left(\frac{k+1}{L}\right) - V\left(\frac{k}{L}\right) - \frac{f}{L}\right)} \\ \text{prob}(k \to k-1) &\equiv p_-(k) = \frac{\delta t}{\delta x^2} T e^{-\frac{1}{2T}\left(V\left(\frac{k-1}{L}\right) - V\left(\frac{k}{L}\right) + \frac{f}{L}\right)} \end{aligned}}$$

And the probability to stay in k is $1 - p_+(k) - p_-(k)$

⇒ always chose δt so that this probability is > 0 ⚠

This choice is justified by the fact that the master equation for $\mathring{P}(k,t)$

$$\mathring{P}(k, t+\delta t) = p_+(k-1)\mathring{P}(k-1,t) + p_-(k+1)\mathring{P}(k+1,t) + (1 - p_+(k) - p_-(k))\mathring{P}(k,t) -$$

gives, in the large L limit, for $P(x,t) = \frac{1}{\delta x} \mathring{P}(\frac{k}{L}, t)$, with $x = \frac{k}{L}$, the FP equat

$$\partial_t P(x,t) = \partial_x ((V'(x) - f) P(x,t)) + T \partial_x^2 P(x,t) \quad [\underline{\underline{\text{CHECK THIS}}}]$$

. Q2a- simulate the particle on a large periodic lattice with the rates above for a potential $V(x) = \cos(2\pi x p)$ as in part 1

Q2b- measure the mean velocity $\bar{v}(f)$ for different temperatures observing the crossover from low to high velocity

Q2c- compare different curves of $\bar{v}(f)$ ———— to the analytical prediction

Exercice 1

3 - Other choice of simulation: continuous space, discrete time

- We have seen in the lecture that the Langevin equation

$$\partial_t x = -V'(x) + f + \eta(t) \quad \text{with} \quad \langle \eta(t)\eta(t')\rangle = 2T\,\delta(t-t')$$

implies, in discrete time,

$$\boxed{x_{t+\delta t} = x_t - V'(x_t)\delta t + f\,\delta t + \mathring{\eta}_t} \qquad \langle \mathring{\eta}_t \mathring{\eta}_{t'} \rangle = 2T\underset{\substack{\uparrow \\ \text{time step}}}{\delta t}\ \underset{\substack{\uparrow \\ \text{Kronecker delta}}}{\delta_{tt'}}$$

with $\mathring{\eta}_t$ a noise of gaussian distribution $\boxed{P(\mathring{\eta}) = \frac{1}{\sqrt{4\pi T\delta t}}\, e^{-\frac{1}{2}\frac{\mathring{\eta}^2}{2T\delta t}}}$ ⊛

- You can thus simulate the equation in discrete time and continuous space $x_t \in [0,1]$ (using periodic boundary conditions) by drawing at each time step a value of the noise $\mathring{\eta}_t$ distributed with ⊛.

- To measure the velocity, keep track of Q_t which is the same as x_t expepted that it is not taken modulo 1.

You can then evaluate as: $\bar{v} = \frac{1}{t}\langle Q_t \rangle$

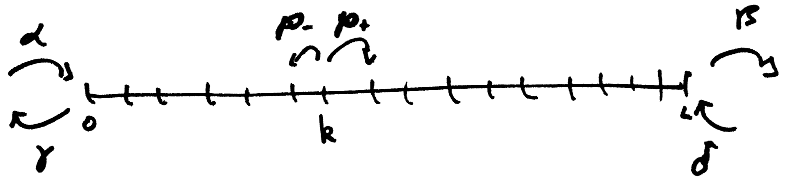
in the large time limit

average over many runs of the simulation

4 - Other situation (Bonus): open system $F(x) = -V'(x)$

$$\partial_t P = \partial_x (V'(x) P(x,t)) + T\partial_x^2 P$$

system driven by boundaries. Corresponds to the microscopic evolution



$$p_\pm(k) = \frac{\delta t}{\delta x^2} T e^{-\frac{1}{2T}\left(V\left(\frac{k\pm1}{L}\right) - V(k)\right)}$$

You can try to answer the same questions as in the periodic case.

Ref: See part 1.1 of J.Tailleur J Kurchan VL J. Phys. A 41 505 001 (2008)