Theoretical Analysis of Complex Systems -
Lecture 1 -

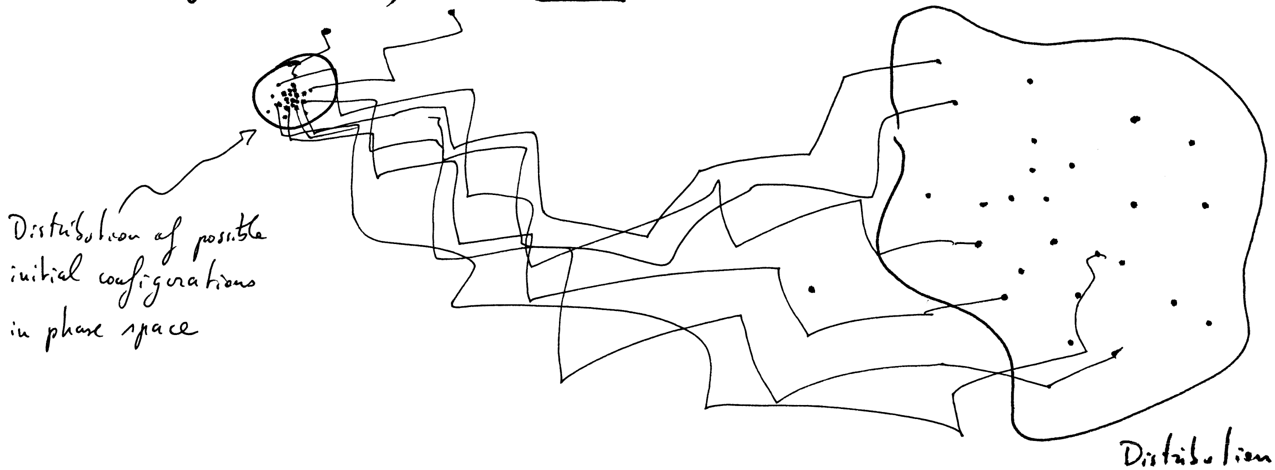
INTRODUCTION TO STOCHASTIC PROCESSES

1 - From microscopic to macroscopic: role of fluctuations and randomness -

- Microscopic classical system: N particles of positions $\{\vec{x}_i^{\,mic}\}$ $1 \leq i \leq N$
  - × Evolution in time: Newton's law $m_i \ddot{\vec{x}}_i^{\,mic} = \underbrace{\vec{F}_i(\{\vec{x}_j^{\,mic}\})}$

    forces ruling interactions between particles and with the environment

    Deterministic equations
  - × Very difficult to solve in general. Yet one would like to understand how things work!
  - × Even if a solution is available
    - high sensitivity to initial conditions (Chaos) (Think of billiards or of the baker transformation)
    - high sensitivity to noise



Distribution of configurations after some time t.

$\underbrace{\hspace{6cm}}$ possible evolutions (each trajectory represents the evolution of all $\{\vec{x}_i^{\,mic}\}$) depending on initial conditions and/or different noise realisations.

→ **Idea**: Focus on distributions rather than on individual trajectories (which also helps to smooth out microscopic details)

- **1st solution**: equilibrium thermodynamics (Boltzmann, Gibbs)
  - × microcanonical ensemble: all configurations with the same energy are equiprobable (or other global conserved quantity)
  - × canonical ensemble: equilibration with a thermostat of temperature β^{-1}

$$\text{Prob}(\text{configuration}) \propto \exp\{-\beta \,\text{Energy}(\text{configuration})\}$$

↳ Applies
- × to determine the mean value of observables
- × in the limit of very large system size
- × at equilibrium (no currents) and for static properties

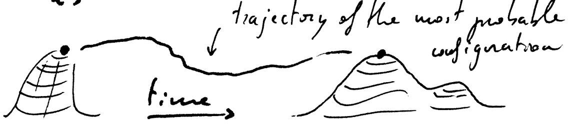
This provide a very generic framework, but is unsuitable to catch
- × full distribution of observables (not only the mean value)
- × non-equilibrium (currents or time evolutions) dynamics
- × generic effects of finite system size

Those effects are less generic (they depend one some microscopic features)

- **2nd solution**: adopt a mesoscopic point of view (i.e. including fluctuations)

Focus one some mesoscopic degrees of freedom $\{\vec{x}_i\}$
Two complementary descriptions:

- × evolution of the distribution $P(\{\vec{x}_i\}, t)$

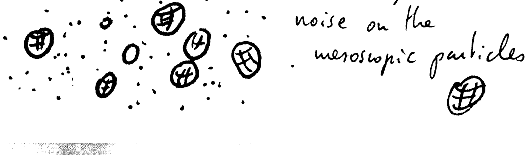


- × (effective) evolution of stochastic trajectories

e.g. $$\boxed{m_i \ddot{\vec{x}}_i = \underbrace{-\gamma_i \dot{\vec{x}}_i}_{\text{friction}} + \underline{\vec{F}_i(\{\vec{x}_j\}, t)} + \underbrace{\vec{\eta}_i}_{\text{noise}}}$$

Both effectively accounting for the effect of microscopic degrees of freedom

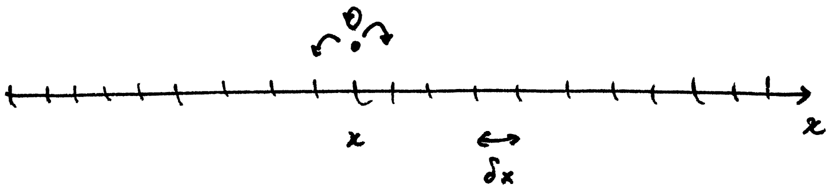
→ traditional example: particles in a fluid
However: much more general framework



2 - Archetypal Example : Random Walk and Diffusion in 1D

Between t and $t+\delta t$ a particle can jump symmetrically to its left or its right on a lattice, or stay at the same place

$\boxed{\mathring{D} \leq 1/2}$



↷ with prob $\mathring{D}$
↶ — $\mathring{D}$
○ — $1-2\mathring{D}$

sum = 1 : conservat° of probability

- **Point of view of trajectories** :

× initial condition: $x_0 = 0$

× evolution: $\boxed{x_{t+\delta t} = x_t + \mathring{\eta}_t}$, noise $\mathring{\eta}_t = \begin{cases} +\delta x & \text{with prob. } \mathring{D} \\ -\delta x & - \quad \mathring{D} \\ 0 & - \quad 1-2\mathring{D} \end{cases}$

Each occurrence of the noise $\mathring{\eta}_t$ is independent of the previous and following ones

This describes the distribution of probability of the noise (The distribution does not depend on time)

How can one take a continuum space limit? $(\delta t \to 0, \delta x \to 0)$
This is not a trivial question :

$$\frac{x_{t+\delta t} - x_t}{\delta t} = \frac{\mathring{\eta}_t}{\delta t}$$

→ depending on how $\delta t \to 0$ & $\delta x \to 0$ it is not obvious to catch how $\mathring{\eta}$ (i.e. $\mathring{D}$) should be tuned so as the limit to be well-defined (i.e. not vanishing nor exploding)

⚠

- **Point of view of distributions** :

$\mathring{P}(x,t) = \underbrace{\text{probability}}_{\text{over all possible realizations of the noise } \mathring{\eta}}$ that the particle is in x at time t

⚠ x in $\mathring{P}(x,t)$ is a dummy variable (do not write $\mathring{P}(x_t)$)

× normalisation : $\forall t, \quad \sum_x \mathring{P}(x,t) = 1$

× initial condition: $\mathring{P}(x,0) = \delta_{x0} = \begin{cases} 1 & x=0 \\ 0 & \text{otherwise} \end{cases}$

× evolution :

"Master equation"

$$\boxed{\mathring{P}(x,t+\delta t) = \underbrace{\mathring{D}\,\mathring{P}(x-\delta x,t) + \mathring{D}\,\mathring{P}(x+\delta x,t)}_{\text{one was in } x\mp\delta x \text{ at time } t} + \underbrace{(1-2\mathring{D})\,\mathring{P}(x,t)}_{\text{one was already in x at time } t.}}$$

- The continuum limit is now easier to tackle: the master equation rewrites:

$$\frac{\mathring{P}(x,t+\delta t)-\mathring{P}(x,t)}{\delta t} = \frac{\mathring{D}}{\delta t}\left[\mathring{P}(x-\delta x,t)+\mathring{P}(x+\delta x,t)-2\mathring{P}(x,t)\right] \quad (*)$$

One introduces a probability **density** $P(x,t)$ of the continuous variables x,t

$$\boxed{P(x,t)=\frac{1}{\delta x}\mathring{P}(x,t)} \quad \text{in the limit } \delta x \to 0,\ \delta t \to 0$$

with normalisation condition $\int dx\, P(x,t) \overset{\forall t}{=} 1$ (arising from $\sum_x \overbrace{\mathring{P}(x,t)}^{\delta x . P(x,t)} = 1$)

and initial condition $P(x,0)=\delta(x)$ (Dirac delta; arising from $P(x,0)=\lim_{\delta x\to 0}\frac{\delta_{0x}}{\delta x}=\delta(x)$)

[Remark: normalizing probabilities is very important!]

Expanding $(*)$ for small δt and small δx one obtains:

$$\partial_t P(x,t) = \frac{\mathring{D}}{\delta t}\Big[\underbrace{P(x,t)+P(x,t)-2P(x,t)}_{=0\ \text{(conservation of probability)}}\ \underbrace{+(-\partial_x P+\partial_x P)\delta x}_{=0\ \text{(symmetry of the noise)}}\ \underbrace{+\left(\tfrac{1}{2}+\tfrac{1}{2}\right)\delta x^2\,\partial_x^2 P(x,t)}_{\text{first non-trivial term of the expansion}}\Big]$$

$$\partial_t P(x,t) = \frac{\mathring{D}\,\delta x^2}{\delta t}\,\partial_x^2 P(x,t)$$

One thus obtains a non-trivial and well defined limit provided $\boxed{D=\frac{\mathring{D}\,\delta x^2}{\delta t}}$ is finite

$$\boxed{\partial_t P(x,t) = D\,\partial_x^2 P(x,t)} \quad (**)$$

This is the **diffusion equation**. (Occurs in a lot of contexts!)

Solution by scaling: $P(x,t) \overset{?}{=} Dt^{-\zeta}\hat{P}(xt^{-\zeta})$ with $1=\int dx\,P(x,t)=\int d(xt^{-\zeta})\,\hat{P}(xt^{-\zeta}) = \int d\hat{x}\,\underset{\hat{P}\text{ normalized to 1}}{\hat{P}(\hat{x})}$

with $\hat{x}=xt^{-\zeta}$, $(**) \Leftrightarrow -\zeta t^{-1-\zeta}\left(\hat{P}(\hat{x})+\hat{x}\hat{P}'(\hat{x})\right) = Dt^{-3\zeta}\hat{P}''(\hat{x})$

meaningful only provided $-1-\zeta=-3\zeta$ ie $\boxed{\zeta=1/2}$

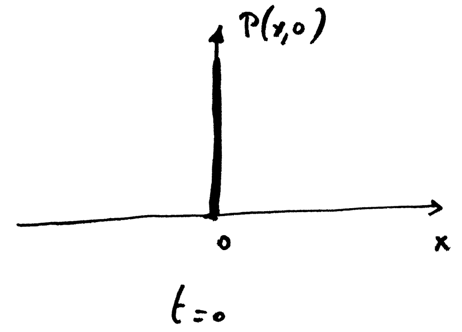
The Partial Differential Equation (PDE) then reduces to an Ordinary Differential Equation (ODE)

$$\hat{P}(\hat{x})+\hat{x}\hat{P}'(\hat{x})+2D\hat{P}''(\hat{x})=0$$

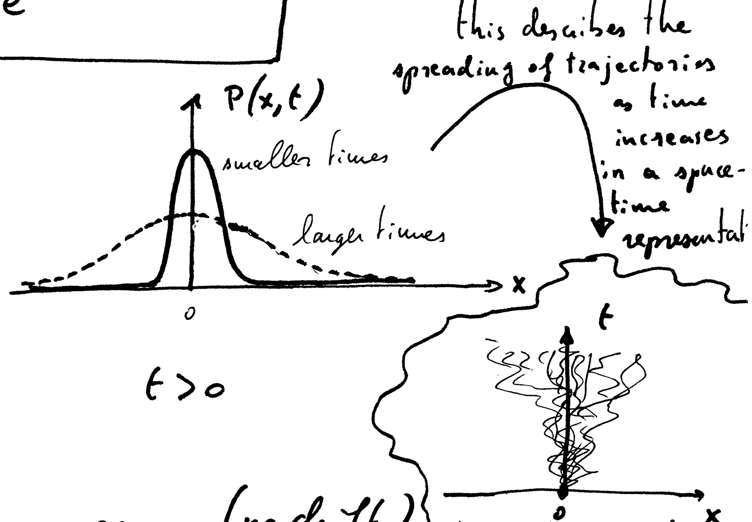
A (normalized) solution is $\boxed{\hat{P}(\hat{x})=\frac{1}{\sqrt{4\pi D}}e^{-\frac{1}{2}\frac{\hat{x}^2}{2D}}}$

3_ Analysis of the solution and continuum limit for trajectories :

One thus finds $\boxed{P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{1}{2}\frac{x^2}{2Dt}}}$



$\xrightarrow[t>0]{\sim}$



this describes the spreading of trajectories as time increases in a space-time representati

set of trajectories x_t for different realizations of the noise

The average $\langle x(t)\rangle = 0$ is always zero (no drift)

The variance $\langle x^2(t)\rangle = 2Dt$ is increasing linearly with time (diffusion)

What is the continuum limit of the noise ?

Let us recall the discrete evolution

$$\frac{x_{t+\delta t} - x_t}{\delta t} = \frac{\mathring{\eta}_t}{\delta t} \qquad \text{with} \qquad \mathring{\eta}_t = \begin{cases} +\delta x & \text{prob. } \mathring{D} \\ -\delta x & \text{prob. } \mathring{D} \\ 0 & \text{prob. } 1-2\mathring{D} \end{cases}$$

one has found that : keeping $D = \frac{\mathring{D}\delta x^2}{\delta t}$ finite is the "correct" continuum limit.
Our aim is to give a meaning to the equation $\boxed{\partial_t x = \eta(t)}$

× average of the noise : where $\eta(t) = \frac{\mathring{\eta}_t}{\delta t}$ the "correct" limit of the noise.

$$\langle \eta(t)\rangle = \langle \partial_t x\rangle = \partial_t \langle x\rangle = \partial_t \int dx\, x\, P(x,t) = 0$$

jump from averaging on trajectories to averaging with the distribution $P(x,t)$

Hopefully, this is compatible with the discrete-time

$$\langle \mathring{\eta}_t\rangle = \delta x \mathring{D} - \delta x \mathring{D} + 0 \times (1-2\mathring{D}) = 0$$

× variance of the noise : determining $\langle \eta(t_1)\eta(t_2)\rangle$ would require to know the two-point distribution $P(x_2 t_2 ; x_1 t_1)$.

One has in fact : $\boxed{P(x_2 t_2 | x_1 t_1) = \frac{1}{\sqrt{4\pi D(t_2-t_1)}} e^{-\frac{1}{2}\frac{(x_2-x_1)^2}{2D(t_2-t_1)}}}$ for $t_2 > t_1$

Let's first determine the correlator $C_2(t_1,t_2) = \langle (x(t_2)-x(t_1))^2\rangle$ of the integral $x(t) = \int_0^t d\tau\, \eta(\tau)$ which is a more regular quantity than $\eta(t)$.

[Indeed determining $\langle \eta(t_1)\eta(t_2)\rangle$ directly may lead to inconsistencies.]

One has: for $t_2 > t_1$

$$C_2(t_1,t_2) = \langle (x(t_2)-x(t_1))^2\rangle = \int dx_1 dx_2 \, (x_2-x_1)^2 \underbrace{P(x_2\, t_2 \,|\, x_1\, t_1)} \, \underbrace{P(x_1\, t_1)}$$

allows to sample the position $x_2 = x(t_2)$ at time $t_2 > t_1$, <u>knowing</u> that one had $x(t_1) = x_1$

allows to follow the evolution of the distribution up to time t_1 and to sample $x_1 = x(t_1)$

The computation is a matter of Gaussian integrals. One finds:

$C_2(t_1,t_2) = 2D\,(t_2-t_1)$ and similarly if $t_1 > t_2$ $C(t_1,t_2) = 2D(t_1-t_2)$

Finally: $\boxed{C_2(t_1,t_2) = \langle (x(t_2)-x(t_1))^2\rangle = 2D\,\underbrace{|t_2-t_1|}}$ ⊛

This absolute value is (although unnoticeably) extremely important.

- **Formal Remark:** link between $\underbrace{\langle (x(t_2)-x(t_1))^2\rangle}_{\equiv C_2(t_1,t_2) \equiv C(t_2-t_1)}$ and $\underbrace{\langle \eta(t_2)\eta(t_1)\rangle}_{\equiv R_2(t_1,t_2)\equiv R(t_2-t_1)}$:

  Using $x(t)-x(0) = \int_0^t dt' \eta(t')$

  One has: $C''(t) = \partial_t^2 \langle [x(t)-x(0)]^2\rangle = \partial_t^2 \int_0^t dt_1 \int_0^t dt_2 \underbrace{\langle \eta(t_2)\eta(t_1)\rangle}_{R(t_2-t_1)} = \dots = R(t) + R(-t)$

  Thus, using the parity $R(t) = R(-t)$ [arising from $\langle \eta(t)\eta(0)\rangle = \langle \eta(0)\eta(-t)\rangle$ by invariance of the distribution of η by translation along time]

  One finally gets: $\boxed{R(t) = \frac{1}{2} C''(t)}$ i.e $\boxed{\langle \eta(t)\eta(0)\rangle = \partial_t^2 \langle (x(t)-x(0))^2\rangle}$

- **Finally:** one gets the variance of $\eta(t)$ from ⊛

$$\boxed{\langle \eta(t_2)\,\eta(t_1)\rangle = 2D\,\delta(t_2-t_1)}$$

Dirac - delta correlations

- **Remark:** <u>this corresponds to a naive argument</u> on $\mathring{\eta}_t$: (with $\eta(t) = \frac{\mathring{\eta}_t}{\delta t}$)

  indiscrete time

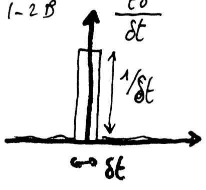
  if $t_2 \neq t_1$, $\langle \eta(t_1)\eta(t_2)\rangle = \frac{1}{\delta t^2}\langle \mathring{\eta}_{t_1}\mathring{\eta}_{t_2}\rangle = 0$ since for $t_1 \neq t_2$, $\mathring{\eta}_{t_1}$ and $\mathring{\eta}_{t_2}$ are uncorrelated

  if $t_1 = t_2 = t$, $\langle \eta(t)\eta(t)\rangle = \frac{1}{\delta t^2}\langle \mathring{\eta}_t^2\rangle = \frac{1}{\delta t^2}\underbrace{\left(\delta x^2 \mathring{D} + \delta x^2 \mathring{D} + 0\cdot(1-2\mathring{D})\right)}$

  since $\mathring{\eta}_t = \begin{cases} \delta x & \text{with prob. } \mathring{D} \\ -\delta x & = \mathring{D} \\ 0 & = 1-2\mathring{D} \end{cases}$

  $= 2\,\frac{\mathring{D}\delta x^2}{\delta t}\,\frac{1}{\delta t}$ where $D = \frac{\mathring{D}\delta x^2}{\delta t}$

  Hence: $\langle \eta(t_1)\eta(t_2)\rangle = 2D\,\frac{\delta_{t_1 t_2}}{\delta t} \xrightarrow[\delta t \to 0]{} 2D\,\delta(t_2-t_1)$



• **Remark** : regularity of the function $x(t)$

One has $\left\langle (x(t+\delta t) - x(t))^2 \right\rangle = \delta t$

Hence $\boxed{|x(t+\delta t) - x(t)| \sim \sqrt{\delta t}}$ which is much larger than δt (one would have $x(t+\delta t)-x(t) \sim \delta t$ for a differentiable / regular-enough funct°)

⚠ Note that from the discrete equation $x_{t+\delta t} = x_t + \eta_t$ this behavior is not obvious as long as one has not determined the correct $\underset{\delta t \to 0}{\delta x \to 0}$ limit.

◐ - the evolution of the distribution is very simple (a Gaussian)
- the trajectories are non-trivial 'processes'.

4. Determination of the distribution of trajectories: the Brownian motion

• After having determined $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t') \rangle = 2D\,\delta(t-t')$

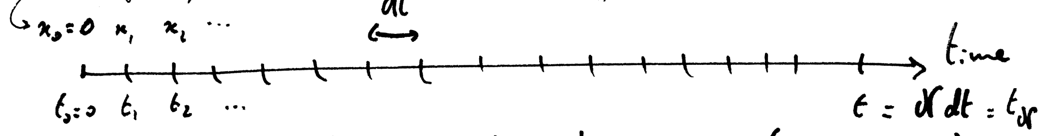
or equivalently $\langle x(t) \rangle = 0$, $\langle [x(t)-x(t')]^2 \rangle = 2D\,|t-t'|$) through $x(t) = \int_0^t dt' \eta(t')$

let us now determine the full distribution of the processes $[\eta(t')]_{0\le t' \le t}$, $[x(t')]_{0 \le t' \le t}$

• This is not direct, since $P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{1}{2}\frac{x^2}{2Dt}}$ gives only the distribution at final time.

• Let us discretize time, keeping x continuous

values of the positions $x_k = x(t_k)$ at different times t_k.



$(dt \gg \delta t)$

$dt = \frac{t}{N}$ $\quad t_k = k\,dt$

One uses $P(x_k t_k | x_{k-1} t_{k-1}) = \frac{1}{\sqrt{4\pi D dt}} \exp\left(-\frac{1}{2}\frac{(x_k - x_{k-1})^2}{2D dt}\right)$

To write $\boxed{P(x_N t_N | x_{N-1} t_{N-1} | \dots | x_0 t_0) = (4\pi D dt)^{\frac{N}{2}} \exp\left(-\frac{1}{2}\sum_{k=1}^{N} \frac{(x_k - x_{k-1})^2}{2D dt}\right)}$

This represents the probability (density) of the history $x_0 t_0 \dots x_N t_N$, allowing to compute any mean-value of observable $\mathcal{O}(x_0 \dots x_N)$ as

$$\langle O(x_0\cdots x_N)\rangle = \int dx_0\cdots dx_N\, O(x_0\cdots x_N)\, P(x_N\, t_N | x_{N-1}\, t_{N-1} | \cdots | x_0\, t_0)\, \underbrace{P_0(x_0)}_{\text{(arbitrary) initial distribution at time } t_0}$$

Going to continuum notations, $\sum_{k=1}^{N} \frac{(x_k-x_{k-1})^2}{2D\,dt} = \sum_{k=1}^{N} dt\, \frac{1}{2D}\left(\frac{x_k-x_{k-1}}{dt}\right)^2 \underset{dt\to 0}{\simeq} \int_0^t d\tau\, \frac{(\partial_\tau x)^2}{2D}$

square braquets ↓

and denoting : Prob[x] the probability density of the history $[x(\tau)]_{0\le\tau\le t}$

$O[x]$ the observable depending on the whole history

$\mathcal{D}x = \frac{dx_0\cdots dx_N}{(4\pi D\,dt)^{N/2}}$ the precise weighted measure on trajectories

note : X = x
sorry for the variation of notation

One may write: Represents an integration over trajectories.

$$\boxed{\begin{aligned} \langle O[x]\rangle &= \int \mathcal{D}x\; O[x]\; \mathrm{Prob}[X]\; P_0(x(0)) \\ \mathrm{Prob}(X) &= \exp\left\{-\frac{1}{2}\int_0^t d\tau\, \frac{(\partial_\tau x)^2}{2D}\right\} \end{aligned}}$$

→ Note that the normalization is hidden in $\mathcal{D}x$ through $\int \mathcal{D}x\; \mathrm{Prob}[X]\, P_0(x(0)) = 1$

- With the notation above, $\langle 1\rangle = 1$: the integral $\int \mathcal{D}x$ is normalized.

Remark : in most cases, this integration over trajectories does not depend on the number of time slices N as $N\to\infty$. In case of doubt: check explicitly!

In the limit $N\to\infty$, the process $[x(\tau)]_{0\le\tau\le t}$ is called a Brownian motion.

- it has zero mean and variance $\langle [x(t_2)-x(t_1)]^2\rangle = 2D\,|t_2-t_1|$
- it has a Gaussian distribution

- In terms of η : the white noise   in discrete: $x_{k+1}-x_k = \eta_k\, dt$

Similarly :

$$\boxed{\begin{aligned} &\text{in discrete time } P(\eta_N\, t_N|\cdots|\eta_0\, t_0) = \exp\left(-\frac{1}{2}\frac{1}{2D}\sum_{k=0}^{N} dt\, \eta_k^2\right)\times (4\pi D\,dt)^{\frac{N+1}{2}} \\ &\text{in continuous time } \mathrm{Prob}[\eta] = \exp\left(-\frac{1}{2}\frac{1}{2D}\int_0^t d\tau\, \eta^2(\tau)\right) \end{aligned}}$$

$\eta(\tau)$ has zero mean and variance $\langle \eta(t_1)\eta(t_2)\rangle = 2D\,\delta(t_2-t_1)$

Its distribution is Gaussian.

Rq : note the difference of indexation. The amount of information in $(x_N \cdots x_0)$ is the same as in $(\eta_{N-1}\cdots \eta_0 ; x_0)$.

Remark: the writing

$$P(\eta_N t_N | \dots | \eta_0 t_0) = \prod_{k=0}^{N} \frac{1}{\sqrt{4\pi D dt}} \, e^{-\frac{1}{2}\frac{\eta_k^2}{2Ddt}}$$

makes it clear that the η_i's are independent and uncorrelated -

Remark: the generalisation to other correlators writes:

× for a $N \times N$ invertible matrix, if: A with $A^T = A$

$$P(\eta_0 \cdots \eta_N) = \sqrt{\det \frac{A}{2\pi}} \; e^{-\frac{1}{2}\eta A \eta}$$

(For η, $A = \frac{1}{2D}\mathbb{1}$, where $\mathbb{1}$ is the identity matrix)

then $\langle \eta_i \eta_j \rangle = (A^{-1})_{ij}$

× for a functional operator $\mathcal{A}(t_1,t_2)$: (which is symmetric: $\mathcal{A}(t_1,t_2) = \mathcal{A}(t_2,t_1)$)

$$\mathrm{Prob}[\eta] = \exp\left(-\frac{1}{2}\int dt_1 dt_2 \; \eta(t_1)\mathcal{A}(t_1,t_2)\eta(t_2)\right)$$

$$\langle \eta(t_1)\eta(t_2) \rangle = \mathcal{A}^{-1}(t_1,t_2)$$

$\mathcal{A}^{-1}$ = functional inverse of $\mathcal{A}$

$$\int dt \;\; \mathcal{A}^{-1}(t_1,t)\,\mathcal{A}(t,t_2) = \delta(t_1-t_2)$$

5_ Physical interpretation and generalizations

- Overdamped dynamics: $\dot{x} = \eta$ rewrites as

$$\underbrace{m\ddot{x}}_{\text{with zero mass}} = \underbrace{-\gamma\dot{x}}_{\text{friction}} \; \underbrace{-V'(x)}_{\text{and no potential}} \; \underbrace{+\eta(t)}_{\text{noise}}$$

with friction coefficient $\gamma = 1$

As a first generalization, one may consider the overdamped Langevin dynamics

$$\boxed{\dot{x} = -V'(x) + \eta(t)}$$

$V(x)$ being a potential of corresponding force $-V'(x)$

in details: $\partial_t x(t) = -V'(x(t)) + \eta(t)$

or: $\boxed{\dot{x} = F(x) + \eta(t)}$ for a generic force

- Equation of evolution for the probability density $P(x,t)$:

$$\partial_t x(t) = \overbrace{F(x(t),t)}^{\text{" deterministic force "}} + \overbrace{\eta(t)}^{\text{"stochastic force"}}$$

$F(x,t)$ is assumed to be regular enough

$$\Big\downarrow \int_t^{t+\delta t}$$

$$\int_t^{t+\delta t} d\tau\, F(x(\tau),\tau) \simeq \delta t\, F(x(t),t)$$

$$\mathring{\eta}_t = \int_t^{t+\delta t} d\tau\, \eta(\tau) = B(t+\delta t) - B(t) \qquad B(t) = \int_0^t d\tau\, \eta(\tau) \text{ : Brownian motion}$$

we have seen that $\langle \mathring{\eta}_t \mathring{\eta}_t \rangle = \langle (B(t+\delta t) - B(t))^2 \rangle = C(\delta t) = \underline{\underline{2D\,\delta t}}$ ⊛

↳ (!) when expanding, $\mathring{\eta}_t$ is of order $\sqrt{\delta t}$.

$$\boxed{x_{t+\delta t} - x_t = \delta t\, F(x_t,t) + \mathring{\eta}_t}$$

Remark: one could take for $\mathring{\eta}_t$: $\begin{cases} +\delta x & \text{prob } \mathring{D} \\ -\delta x & \mathring{D} \\ 0 & 1-2\mathring{D} \end{cases}$

As for the free case ($F=0$) one could take the limit $\delta x \to 0$, $\delta t \to 0$, $\frac{\delta x^2}{\delta t}$ fixed

× **1st try: directly expand $P(x,t+\delta t)$**

$$P(x,t+\delta t) = \int dx_1\, P(x_1,t)\, P(x,t+\delta t \,|\, x_1,t)$$

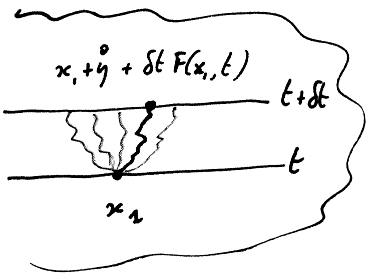
$$P(x,t+\delta t) - P(x,t) = \int dx_1\, P(x_1,t) \Big[\underbrace{P(x,t+\delta t \,|\, x_1,t) - \delta(x-x_1)}_{} \Big]$$

a small δt expansion would be difficult
(it is difficult to expand around a Dirac delta)
⇒ in such situations, integrate over a "test function":

× **2nd try: consider the time-evolution of the average of an observable $\varphi(x)$:**

Consider an observable $\varphi(x)$, vanishing at $|x| \to \infty$ (a "test function", for mathematicians)

$$\langle \varphi(x) \rangle_{t+\delta t} = \int \overset{\text{possible positions at time } t}{dx_1} \underbrace{\int d\mathring{\eta}\, P[\mathring{\eta}]}_{\text{average over the noise } \mathring{\eta}} \underbrace{P(x_1,t)}_{\text{distribution at time } t} \underbrace{\varphi\Big(x_1 + \overset{\text{"of order } \sqrt{\delta t}\text{"}}{\mathring{\eta}} + \delta t\, F(x_1,t)\Big)}_{\text{the expansion at small } \delta t \text{ is now easier}}$$



$\langle \mathring{\eta} \rangle = 0$, $\langle \mathring{\eta}^2 \rangle = 2D\delta t$
see ⊛

$$\simeq \varphi(x_1) + \mathring{\eta}\, \varphi'(x_1) + \delta t\, F(x_1,t)\, \varphi'(x_1) + \frac{1}{2} \underbrace{\mathring{\eta}^2}_{} \varphi''(x_1) + \mathcal{O}(\delta t^{3/2})$$

of order δt after averaging, see ⊛

$$\langle \varphi(x) \rangle_{t+\delta t} = \int dx_1\, P(x_1,t) \Big\{ \varphi(x_1) + \delta t\, F(x_1,t)\, \varphi'(x_1) + D\, \delta t\, \varphi''(x_1) \Big\} + \mathcal{O}(\delta t^{3/2})$$

$$\langle \varphi(x) \rangle_{t+\delta t} = \langle \varphi(x) \rangle_t + \delta t \Big\langle F(x,t)\, \varphi'(x) + D\, \varphi''(x) \Big\rangle_t + \mathcal{O}(\delta t^{3/2})$$

One can now take the limit $\delta t \to 0$ by isolating $\dfrac{\langle \varphi(x)\rangle_{t+\delta t} - \langle\varphi(x)\rangle_t}{\delta t}$

which yields
$$\boxed{\partial_t \langle \varphi(x)\rangle_t = \left\langle F(x,t)\,\varphi'(x) + D\,\varphi''(x)\right\rangle_t} \quad ⊛$$

Remark: from the Langevin equation $\partial_t x = F(x,t) + \eta$

one is tempted to write $\partial_t \varphi(x_t) = \partial_t x_t\, \varphi'(x_t) = (F(x_t,t) + \eta_t)\,\varphi'(x_t)$

and to average as
$$\partial_t \langle\varphi(x)\rangle_t = \Big\langle \underbrace{F(x,t)\,\varphi'(x)}_{\text{"deterministic term"}} + \underbrace{\eta\ \varphi'(x)}_{\text{"stochastic term"}} \Big\rangle$$

the "deterministic term" is the first term of ⊛ and is easy to see.

the "stochastic term" $\langle \eta\varphi'(x)\rangle$ is not easy to determine and is equal to $D\langle\varphi''\rangle_t$ (see ⊛).

"white noise"

→ this is due to the fact that $\eta(t)$ is a very irregular function

Remark: ⊛ is also true without averaging.

This is the Itō formula
$$\boxed{\partial_t[\varphi(\overset{x(t)}{x},t)] = \partial_t\varphi(x,t) + F(x,t)\,\partial_x\varphi(x,t) + D\,\partial_x^2\varphi(x,t)}$$

Remark: time-derivative of averages: points of view of trajectories and distributions

From the point of view of trajectories: difference of values of the function φ on the same trajectory x_t

$$\left\langle \overbrace{\varphi(x_{t+\delta t}) - \varphi(x_t)} \right\rangle = \underbrace{\int dx\ P(x,t+\delta t)\,\varphi(x)} - \underbrace{\int dx\ P(x,t)\,\varphi(x)}$$

Here x is a dummy variable and represents all possible values of x at time $t+\delta t$, of probability $P(x,t+\delta t)$

same as at time t

$$\langle \varphi(x_{t+\delta t}) - \varphi(x_t)\rangle = \int dx\ [P(x,t+\delta t) - P(x,t)]\,\varphi(x)$$

$$\Big\downarrow \ \delta t \to 0$$

$$\boxed{\underbrace{\partial_t \langle\varphi(x)\rangle_t} = \underbrace{\int dx\ \partial_t P(x,t)\ \varphi(x)}}$$

Viewpoint of trajectories (see computation p 10-11) or viewpoint of distributions, depending on what is useful.

Viewpoint of probability distributions

× Equation on the probability distribution :

$$\forall\varphi \qquad \partial_t \langle \varphi(x) \rangle_t = \langle F(x,t)\,\varphi'(x) + D\,\varphi''(x) \rangle_t$$

$$\int dx\, \partial_t P(x,t)\,\varphi(x) = \int dx \left\{ P(x,t)\,F(x,t)\,\varphi'(x) + D P(x,t)\,\varphi''(x) \right\}$$

integrations by part, using $|\varphi| \xrightarrow[x\to\infty]{} 0$

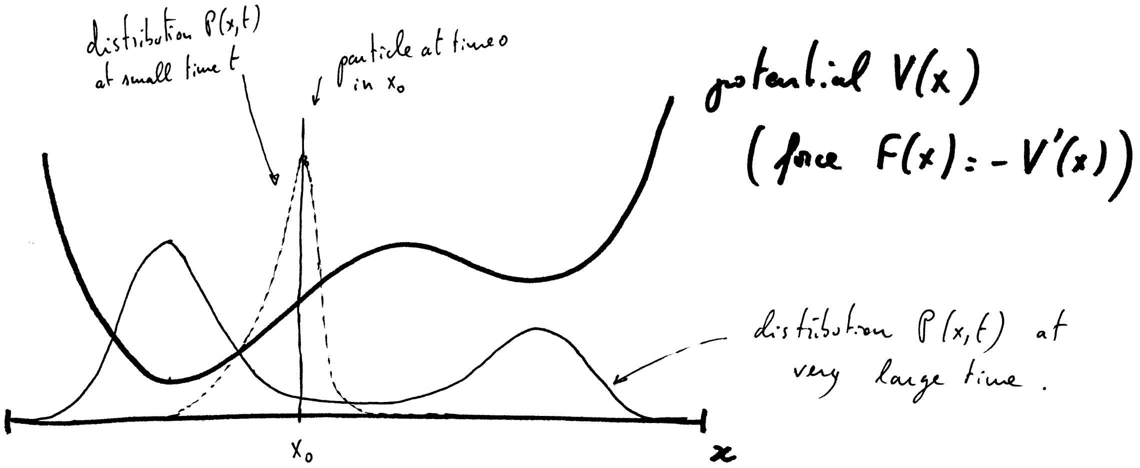
$$\int dx\, \partial_t P(x,t)\,\varphi(x) = \int dx \left\{ -\partial_x\left[P(x,t)F(x,t)\right] + D\,\partial_x^2 P(x,t) \right\} \varphi(x)$$

This equation is valid for all functions $\varphi(x)$ decreasing fast enough at infinity

Hence: $$\boxed{\partial_t P(x,t) = -\partial_x\left[F(x,t)P(x,t)\right] + D\,\partial_x^2 P(x,t)}$$

This is the Fokker-Planck equation for a 1D particle in a force $F(x,t)$.

× Example :



- Steady state: limit distribution $P_{st}(x) = \lim_{t\to\infty} P(x,t)$ if it exists

It verifies $$0 = \partial_x\left[-F(x)P_{st}(x) + D\,\partial_x P_{st}(x)\right]$$ for a time-independent force $F(x,t)$

× Remark : for the free particle $(F=0)$ one has $\partial_t P = D\partial_x^2 P$ (diffusion equation)

. on the real line : $P(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}} \xrightarrow[t\to 0]{} 0$ there is no steady state

[long-time properties are catched here through scaling : $P(x,t) \sim \frac{1}{\sqrt{t}}\hat{P}(x/\sqrt{t})$]

. on an interval $[a,b]$: the steady state verifies $\begin{cases} \partial_x^2 P_{st}(x) = 0 \\ \partial_x P_{st}|_a = \partial_x P_{st}|_b = 0 \end{cases}$: no current on boundaries

(for an isolated system) $$\boxed{P_{st}(x) = \frac{1}{b-a}}$$ uniform distribution

- Equilibrium steady state: in a potential $V(x)$ at temperature T

× remark: if $P_{eq}(x) \propto e^{-\frac{1}{T}V(x)}$ (Boltzmann distribution)

one has $T\,\partial_x P_{eq}(x) = -V'(x)\,P_{eq}(x)$

hence:

$$\boxed{\text{The Boltzman distribution } P_{eq}(x) \propto e^{-\frac{1}{T}V(x)} \text{ is a steady-state solution of the Fokker-Planck equation } \partial_x\underbrace{\left[+V'(x)P_{eq}(x) + T\,\partial_x P_{eq}(x)\right]}_{=0} = 0}$$

This provides a link between thermodynamics (Boltzman canonical ensemble) stochastic dynamics (Langevin dynamics)

where $D = T$ (hence the interpretation of $\eta(t)$ as a thermal noise)

× interpretation in terms of a probability current

The FP equation writes $\partial_t P + \partial_x J[P] = 0$ with $J[P] = FP - D\partial_x P$

A steady-state distribution verifies $\partial_x J(P) = 0$ $\Rightarrow J[P] \equiv j$ (constant) [specific of 1D]

An equilibrium st.st. distribution — $J(P) = 0$ $j = 0$

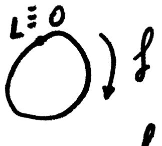
when $j \neq 0$, we speak of a non-equilibrium steady state NESS

× mean velocity in the system: the example of $V = -fx$ i.e. $F(x) = f$

① $\langle \partial_t x \rangle = \langle f + \eta \rangle \Rightarrow \boxed{\langle \partial_t x \rangle \equiv v = f}$

② for distribution and for the NESS:

one considers periodic boundary conditions



$$\langle \partial_t x \rangle = \int dx\; x\,\partial_t P(x,t) = -\int dx\; x\,\partial_x J[P] \quad \text{from Fokker-Planck}$$

$$\overset{ipp}{=} \int dx\; J(P) \qquad \text{but} \qquad J(P) = j \text{ is a constant } j$$

Hence: $\boxed{\text{the velocity is } v \equiv \langle \partial_t x \rangle = j = f}$

The system is in a true NESS $(j \neq 0)$ iff $f \neq 0$.

It is not obvious to determine $P_{st}(x)$ for $f \neq 0$. See exercises.

↳ indeed a try as $P_{st}(x) \propto e^{-\frac{1}{T}fx}$ does not respect boundary conditions

- "*Equilibrium dynamics*" and *reversibility* :

× if there exists a potential $V(x)$ such that $F(x) = -V'(x)$ [independent of time]

and that $P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T}V(x)}$, $Z = \int dx\, e^{-\frac{1}{T}V(x}$

is the stationnary distribution $\left[\underline{\underline{J[P_{eq}]=0}}\right]$

then we say that the dynamics is an "equilibrium dynamics"

and that $P_{eq}(x)$ is an "equilibrium steady state" (or "reversible measure")

× otherwise, the dynamics is a "non-equilibrium dynamics". $\underline{\underline{[J[P_{st}]\neq 0]}}$

→ There is a physical interpretation in terms of reversibility.

Let us fix the final time t and consider all trajectories $x(\tau)$ for $0 \leq \tau \leq t$.

× probability density of a trajectory $(x(\tau))_{0\leq\tau\leq t}$:

simpliest way of obtaining it:

$$\text{Prob}[x(\tau)] = \int \mathcal{D}\eta \; \delta\left(\partial_t x - F(x) - \eta\right) e^{-\frac{1}{4T}\int_0^t d\tau\; \eta(\tau)^2}$$

This is justified e.g. from the expression of the average of an observable depending on $\langle \mathcal{O}[x(\tau)]\rangle = \int \mathcal{D}x\; \mathcal{O}[x(\tau)]\,\text{Prob}[x$ ...

$$\boxed{\text{Prob}[x(\tau)] = \exp\left\{-\frac{1}{4T}\int_0^t d\tau\; \left(\partial_\tau x - F(x)\right)^2\right\}}$$

× time-reversed trajectory: one defines $x^R(\tau) = x(t-\tau)$ $(0 \leq \tau \leq t)$

which verifies $x^R(t) = x(0)$ $x^R(0) = x(t)$ $\partial_\tau x^R_{(t-\tau)} = -\partial_\tau x(\tau)$

× joint probability of history & initial condition in the equilibrium dynamics:

$$P_{eq}(x(0))\,\text{Prob}[x(\tau)] \propto \underbrace{e^{-\frac{1}{T}V(x(0))}}_{=e^{-\frac{1}{T}V(x^R(t))}} \exp\left\{-\frac{1}{4T}\underbrace{\int_0^t d\tau\; \left(\partial_\tau x + V'(x)\right)^2}_{}\right\}$$

$x \mapsto x^R$
$\tau \mapsto t-\tau$

$$\int_0^t d\tau\; \left(-\partial_\tau x^R + V'(x^R)\right)^2 = \int_0^t d\tau \left[\left(\partial_\tau x^R + V'(x^R)\right)^2 - 4\overbrace{\partial_\tau x^R V'(x^R)}^{\partial_\tau(V(x^R))}\right]$$

so as to recover a +

$$= -4\left(V(x^R(t)) - V(x^R(0))\right) + \int_0^t d\tau\left(\partial_\tau x^R + V'(x^R)\right)^2$$

those two terms **cancel**

this one remains

There is an equivalence btw
- $J[P]=0$ (no prob. current)
- $P(x) \propto e^{-\frac{1}{T}V(x)}$
- reversibility in that sense

$$\propto e^{-\frac{1}{T}V(x^R(0))} \exp\left\{-\frac{1}{4T}\int_0^t d\tau\; \left(\partial_\tau x^R + V'(x^R)\right)^2\right\}$$

Hence:

$P_{eq}(x(0))\,\text{Prob}[x(\tau)] = P_{eq}(x^R(0))\,\text{Prob}[x^R(\tau)]$, i.e.

there is reversibility (trajectories and their time-reversed have the same probability, including initial eq. distrib.)

when the dynamics is an "equilibrium dynamics".

otherwise not.

- Reversible dynamics, Doob transform and Schrödinger equation

Consider the "equilibrium dynamics" (also termed "reversible dyn.")

$D = T$ from now on

$$\partial_t x = -V'(x) + \eta \quad \Leftrightarrow \quad \partial_t P(x,t) = +\partial_x\left[V'(x)\,P(x,t)\right] + \overset{\nearrow}{T}\,\partial_x^2 P(x,t)$$

This equation for $P(x,t)$ is linear, first-order in time, and looks like Schrödinger's equation $-i\partial_t \Psi(x,t) = \left[\partial_x^2 - V_{quant}(x)\right]\Psi(x,t)$ on the wave function Ψ without however taking the same form.

Let's introduce $\boxed{P_{sym}(x,t) = e^{\frac{1}{2T}V(x)}\,P(x,t) = P_{eq}^{-1/2}(x)\,P(x,t)}$ (This is the Doob transform)

One easily checks that the Fokker-Planck equation is equivalent to:

$$\boxed{\begin{aligned} \partial_t P_{sym}(x,t) &= \left[T\partial_x^2 - V_{eff}(x)\right] P_{sym}(x,t) \\ V_{eff}(x,t) &= \frac{1}{4T}\left(V'(x)\right)^2 - \frac{1}{2}V''(x) \end{aligned}}$$ effective potential

Hence: the FP equation for P_{sym} is equivalent to a [imaginary time] Schrödinger equation for P_{sym} in a potential $V_{eff}(x)$

Again, this only works for "equilibrium dynamics".
Such a correspondence can be established in a much more general context, see later.

- Remark: correct form of the probability density of trajectories

The (quantum-like) Feynman path-integral would yield for the probability of histories:

$$P_{sym}(x,t) = \frac{1}{Z}\, e^{-\frac{1}{2T}V(x(0))}\; e^{-\frac{1}{4T}\int_0^t d\tau \left\{ \underbrace{(\partial_\tau x)^2 + (V'(x))^2}_{\text{up to time-boundary terms, corresponds to } (\partial_\tau x + V'(x))^2} \underbrace{- 2T V''(x)}_{\text{this term is absent from the expression p. 14}} \right\}}$$

→ This comes from a different choice of time-discretization of the Langevin equation (Itō vs Stratonovitch) → $e^{\int V''}$ can be seen as a Jacobian.

See A.W.C. Lau & T.C. Lubensky PRE 76 011123 (2007) for details

• Generalization to several interacting particles :

$x_i(t)$ each of them in a noise $\eta_i(t)$ $\quad 1 \leq i \leq N$

global interactions $F_i(\{x_j\},t)$

$$\boxed{\begin{aligned}\partial_t x_i &= F_i(\{x_j\},t) + \eta_i(t)\\ \langle \eta_i(t)\,\eta_j(t')\rangle &= 2\,\Delta_{ij}\,\delta(t-t')\end{aligned}}$$

with $\{\eta_j(t)\}$ white noises

Δ_{ij} invertible symmetric matrix

η_j's are uncorrelated if $\Delta_{ij}=D\,\delta_{ij}$

Gaussian distribution of the noise :

$$\text{Prob}[\{\eta_j(t)\}] \propto \exp\left(-\frac{1}{2}\int_0^t d\tau \sum_{ij}\eta_i(\tau)\left((2\Delta)^{-1}\right)_{ij}\eta_j(\tau)\right)$$

if $\Delta_{ij}=\delta_{ij}$

$$\propto \exp\left(-\frac{1}{4D}\int_0^t d\tau \sum_i \eta_i(\tau)^2\right)$$

× Fokker-Planck equation: notation $\partial_i = \partial_{x_i} = \dfrac{\partial}{\partial x_i}$

$$\boxed{\partial_t P(\overbrace{\{x_j\}}^{x},t) = -\underbrace{\sum_i \partial_i\left(F_i(x,t)\,P(x,t)\right)}_{\text{deterministic contribution}} + \underbrace{\sum_{ij}\Delta_{ij}\,\partial_i\partial_j P(x,t)}_{\text{(correlated) diffusive contribut}^{\circ}}}$$

in matrix notations :

$$\partial_t P = -\vec{\nabla}\cdot\left(\vec{F}(x,t)\,P(x,t)\right) + \vec{\nabla}\cdot\left(\overset{\text{matrix } \Delta_{ij}}{\Delta}\,\vec{\nabla}P(x,t)\right)$$

if the noise is uncorrelated $(\Delta_{ij}=D\delta_{ij})$

$$\partial_t P = -\vec{\nabla}\cdot\left(\vec{F}(x,t)\,P(x,t)\right) + D\,\vec{\nabla}^2 P(x,t)$$

× Itô formula :

$$\boxed{\partial_t\varphi(x,t) = \sum_i F_i(x,t)\,\partial_i\varphi(x,t) + \sum_{ij}\Delta_{ij}\,\partial_i\partial_j\varphi(x,t)}$$

× **Equilibrium :**

When $\forall i$, $\underbrace{F_i(x,t)}_{\text{forces acting on particle } i} = -\underbrace{\partial_i V(x)}_{\text{common potential to all particles}}$

↑ describing interactions between particles and also with the environment

i.e. when "forces are conservative"
i.e. when "forces derive from a potential"
then the Boltzmann-Gibbs measure is an equilibrium steady state

① For non-correlated noises $\langle \eta_i \eta_j \rangle = 2D\, \delta_{ij}\, \delta(t-t')$; denoting $\boxed{D=T}$

$$\boxed{P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T}V(x)}} \qquad \text{with } Z = \int dx\; e^{-\frac{1}{T}V(x)}$$

verifies $\partial_i P_{eq}(x) = -\frac{1}{T}\partial_i V(x)\; P_{eq}(x)$ hence $\underline{+\partial_i V P_{eq} + T P_{eq}} = 0$

but the FP equation writes $\partial_t P = +\sum_i \partial_i \big[\underbrace{\partial_i V P + T P} \big]$

hence Peq(x) is an equilibrium distribution

all components of the current are 0

② For correlated noises $\langle \eta_i(t)\eta_j(t')\rangle = 2\Delta_{ij}\,\delta(t-t')$:

$$P_{eq}(x) = \frac{1}{Z} \exp\big(-W(x)\big) \Rightarrow \partial_i P_{eq} = -\partial_i W\, P_{eq}$$

searching for a solution of $\partial_i V P_{eq} + \sum_j \Delta_{ij}\, \partial_j P_{eq} = 0$ is equivalent to

$$\forall i,\ \partial_i V = \sum_j \Delta_{ij}\, \partial_j W \quad \text{ie} \quad \forall i,\ \boxed{\partial_j W = \sum_i (\Delta^{-1})_{ij}\, \partial_j V}$$

A necessary and sufficient condition for this equation to have a solution (using Poincare's lemma) is : $\forall i,j,\ \sum_k (\Delta^{-1})_{ki}\, \partial_j V = \sum_k (\Delta^{-1})_{kj}\, \partial_i V$

A possible solution is for quadratic potentials : $V = \sum_{ij} \Delta_{ij}\, x_i x_j$

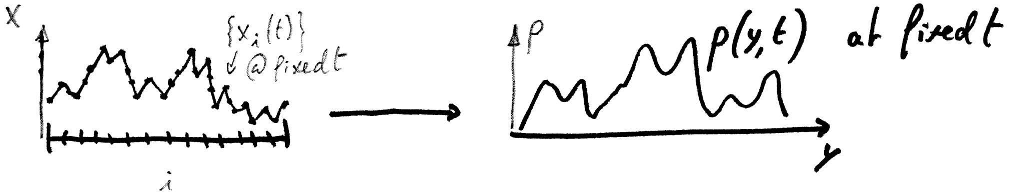
‖ But otherwise there is no generic characterisation of equilibrium with correlated noise.

- Generalisation to several coupled noises: continuum

$x \equiv \rho$
$y \equiv i$

$$x_i(t) \longrightarrow \rho(y,t)$$

$$\eta_i(t) \longrightarrow \eta(y,t)$$



→ one will call $\rho(y,t)$ a "profile", but the description holds in generality

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\Delta_{ij}\,\delta(t.t') \longrightarrow \langle \eta(y,t)\,\eta(y',t')\rangle = \underbrace{R(y'-y)}\,\delta(t-t')$$

describes correlations in direction y

[$R(y) = 2D\,\delta(y'-y)$ for uncorrelated noise]

$$P(x,t) \longrightarrow P[\rho,t]$$ (functional) probability density of the profile $\rho(y)$, at time t

- Langevin equation: from $\partial_t x_i = F_i(x,t) + \eta_i$ (see p. 1.16) one goes to

$$\boxed{\partial_t \rho(y,t) = \mathcal{F}[\rho; y,t] + \eta(y,t)}$$

- Fokker-Planck equation: from $\partial_t P(x,t) = -\sum_i \partial_i\big(F_i(x,t)P(x,t)\big) + \sum_{ij}\Delta_{ij}\,\partial_i\partial_j P(x,t)$ one has

$$\boxed{\partial_t P[\rho,t] = -\int dy\,\frac{\delta}{\delta\rho(y)}\Big(\mathcal{F}[\rho;y,t]\,P[\rho,y]\Big) + \frac{1}{2}\int dy\,dy'\,R(y'-y)\,\frac{\delta^2 P[\rho,t)}{\delta\rho(y)\,\delta\rho(y')}}$$

- Itō formula:

$$\partial_t\big(\varphi[\overset{\rho(y,t)}{\rho},t]\big) = \partial_t\varphi[\rho,t] + \int dy\,\mathcal{F}[\rho;y,t]\,\frac{\delta\varphi[\rho,t]}{\delta\rho(y)} + \frac{1}{2}\int dy\,dy'\,R(y'-y)\,\frac{\delta^2\varphi[\rho,t]}{\delta\rho(y)\,\delta\rho(y')}$$

- Example of "equilibrium (i.e. reversible) dynamics": $\mathcal{F}[\rho;y,t] = \dfrac{\delta \mathcal{H}[\rho]}{\delta\rho(y)}$

Then for an uncorrelated noise $R(y) = 2T\,\delta(y)$

$$\boxed{P_{eq}[\rho] \propto \exp\left(-\frac{1}{T}\mathcal{H}[\rho]\right)}$$ is a solution of the FP equation, with 0 current:

$$-\mathcal{F}[\rho;y,t]\,P_{eq}[\rho] + \frac{1}{2T}\,\frac{\delta P_{eq}[\rho]}{\delta\rho(y)} = 0$$

1.19

Lecture on Complex Systems

2012

- Generalization to position-dependent noise:

But writing:

$D(x,t) > 0$

$$\boxed{\partial_t x = F(x,t) + \sqrt{2D(x,t)}\;\eta(t)}$$ with $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$

Fokker-Planck: $\mathrm{Prob}[x(t)] \propto \exp\left(-\frac{1}{2}\int_0^t d\tau \frac{(\partial_\tau x - F(x,t))^2}{2D(x,t)}\right)$

$$\boxed{\partial_t P(x,t) = -\partial_x\left(F(x,t)P(x,t)\right) + \partial_x^2\left(D(x,t)P(x,t)\right)}$$

- With several coupled noises:

$$\boxed{\partial_t x_i = F_i(\mathbf{x},t) + \sqrt{2D(x,t)}\;\eta_i(t)}$$ $\langle \eta_i(t)\eta_j(t')\rangle = \Delta_{ij}\delta(t-t'$

$$\boxed{\partial_t P(x,t) = -\sum_i \partial_i\left(F_i(x,t)P(x,t)\right) + \sum_{ij}\partial_i\partial_j\left(\Delta_{ij}D(x,t)P(x,t)\right)}$$

- Generalization to fields: $x_i(t) \mapsto \rho(y,t)$

$$\partial_t \rho(y,t) = \mathcal{F}\left(\rho(x,t),t\right) + \sqrt{2D(\rho(y,t))}\;\eta(y,t) \qquad \langle \eta(y,t)\eta(y',t')\rangle = R(y-y')\delta(t-t')$$

$$\partial_t \mathbf{P}[\rho(y),t] = -\int dy \frac{\delta}{\delta\rho(y)}\left(F(\rho(y),t)P[\rho(y),t]\right) + \int dy\,dy' \frac{\delta^2}{\delta\rho(y)\delta\rho(y')}\left[R(y-y')P[\rho,t] \times D(\rho(y))\right]$$

Remark : Backward Fokker-Planck equation:

We have seen that $P(x,t|x_0 t_0) = P(x,t)$ verifies the forward FP equation

$$\partial_t P(x,t) = -\partial_x \left(F(x,t) P(x,t)\right) + D\partial_x^2 P(x,t)$$

We might be interested in the derivatives $\partial_{x_0}, \partial_{t_0}$ w.r.t. the initial conditions (e.g. to tackle 1st passage problems).

As on page 1.10, we integrate the Langevin equation $\dot{x} = +F(x,t) + \eta$ $\delta t > 0$ between $t_0 - \delta t$ and t_0 to write

$$\underbrace{x(t_0)}_{\equiv x_0} = \underbrace{x(t_0-\delta t)}_{\equiv x_{-1}} + \delta t\, F(x_{-1}, t-\delta t) + \underbrace{\overset{\circ}{\eta}}_{\int_{t_0-\delta t}^{t_0} d\tau\, \eta(\tau)}$$

with $\overset{\circ}{\eta}$ Gaussian, $\langle \overset{\circ}{\eta} \rangle = 0$ $\langle \overset{\circ}{\eta}{}^2 \rangle = 2D\delta t$

$$\Downarrow$$

$$x_{-1} = x_0 - \delta t\, \underbrace{F(x_0,t)} - \overset{\circ}{\eta}$$

Then, for any function $f(x)$ one has:

$$\int dx\, f(x,t)\, P(x,t|x_0,t_0) = \int dx\, f(x,t) \int d\overset{\circ}{\eta}\, P(\overset{\circ}{\eta})\, \underbrace{P(x,t\,|\, \overbrace{x_0 - \overset{\circ}{\eta} - \delta t\, F(x_0,t)}, t_0-\delta t)}$$

or $F(x_{-1}, t-\delta t)$ but this is the same at the order δt we are interested in

we perform the average $\int d\overset{\circ}{\eta} P(\overset{\circ}{\eta})$ through $\langle \overset{\circ}{\eta} \rangle = 0$, $\langle \overset{\circ}{\eta}{}^2 \rangle = 2D\delta t$

$$\simeq P(x,t|x_0,t_0-\delta t) - \overset{\circ}{\eta}\, \partial_{x_0} P(x,t|x_0,t_0) - \delta t\, F(x_0,t)\, \partial_{x_0} P(x,t|x_0,t_0) - \frac{1}{2}\overset{\circ}{\eta}{}^2 \partial_{x_0} P(x,t|x_0,t_0) + O(\delta t^{3/2})$$

↑ one has to expand up to order δt

Collecting :

$$\int dx\, f(x,t) \frac{\overbrace{P(x,t|x_0,t_0) - P(x,t|x_0,t_0-\delta t)}^{\xrightarrow{\delta t\to 0} \partial_{t_0} P(x,t|x_0,t_0)}}{\delta t} = \int dx\, f(x,t) \left[-F(x_0,t)\, \partial_{x_0} P(x,t|x_0,t_0) - D\partial_{x_0}^2 P(x,t|x_0,t_0) \right]$$

And finally:

Note the change of sign & the position of ∂_{x_0}

$$\boxed{-\partial_{t_0} P(x,t|x_0,t_0) = F(x_0,t)\, \partial_{x_0} P(x,t|x_0,t_0) + D\partial_{x_0}^2 P(x,t|x_0,t_0)}$$

This is the backward Fokker-Planck equation, which describes the variations of the probability density with respect to its initial conditions. $\nearrow P(x,t|x_0 t_0) = e^{(t-t_0)\mathbb{W}} P_0(x_0)$

↑ indep. of time

For homogeneous processes where $P(x,t|x_0 t_0)$ is a function of $t-t_0$ only, e.g. if $F(x,t) = F(x)$ one has $\partial_{t_0} = -\partial_t$ and hence

$$\boxed{\partial_t P(x,t|x_0,t_0) = F(x_0)\, \partial_{x_0} P(x,t|x_0,t_0) + D\partial_{x_0}^2 P(x,t|x_0,t_0)}$$