- **Equation of evolution for the probability density $P(x,t)$:**

$$\partial_t x(t) = \overbrace{F(x(t),t)}^{\text{"deterministic force"}} + \overbrace{\eta(t)}^{\text{"stochastic force"}}$$

$F(x,t)$ is assumed to be regular enough

$\int_t^{t+\delta t}$

$$\int_t^{t+\delta t} d\tau\, F(x(\tau),\tau) \simeq \delta t\, F(x(t),t)$$

Remark: without forces ($F=0$) this was denoted $x(t)$ previously; now that $x(t)$ evolves with $F(x,t)$, we call this $B(t)$

$\underbrace{}$ $B(t) = \int_0^t d\tau\, \eta(\tau)$: Brownian motion

where we denote by definition $\boxed{\mathring{\eta}_t = \int_t^{t+\delta t} d\tau\, \eta(\tau) = B(t+\delta t) - B(t)}$

we have seen that $\langle \mathring{\eta}_t \mathring{\eta}_t \rangle = \langle (B(t+\delta t) - B(t))^2 \rangle = C(\delta t) = \underline{\underline{2D\,\delta t}}$ (*)

↳ (!) when expanding, $\mathring{\eta}_t$ is of order $\sqrt{\delta t}$.

$$\boxed{x_{t+\delta t} - x_t = \delta t\, F(x_t,t) + \mathring{\eta}_t}$$

Remark: one could take for $\mathring{\eta}_t$: $\begin{cases} +\delta x & \text{prob } \mathring{D} \\ -\delta x & \mathring{D} \\ 0 & 1-2\mathring{D} \end{cases}$

As for the free case ($F=0$) one could take the limit $\begin{smallmatrix}\delta x \to 0 \\ \delta t \to 0\end{smallmatrix}$, $\frac{\delta x^2}{\delta t}$ fixed

× 1st try: directly expand $P(x,t+\delta t)$

$$P(x,t+\delta t) = \int dx_1\, P(x_1,t)\, P(x, t+\delta t \,|\, x_1, t)$$

$$P(x,t+\delta t) - P(x,t) = \int dx_1\, P(x_1,t) \Big[\underbrace{P(x,t+\delta t \,|\, x_1,t) - \delta(x-x_1)}_{} \Big]$$

a small δt expansion would be difficult

(it is difficult to expand around a Dirac delta)

⇒ in such situations, integrate over a "test function":

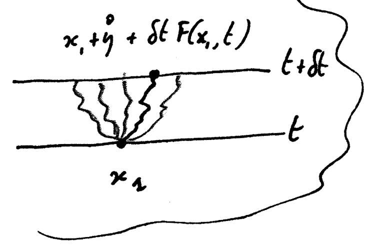
× 2nd try: consider the time-evolution of the average of an observable $\varphi(x)$:

Consider an observable $\varphi(x)$, vanishing at $|x|\to\infty$ (a "test function", for mathematicians)

Average value at time $t+\delta t$

possible positions at time t

$$\langle \varphi(x) \rangle_{t+\delta t} = \int dx_1 \underbrace{\int d\mathring{\eta}\, P[\mathring{\eta}]}_{\text{average over the noise } \mathring{\eta}} \underbrace{P(x_1,t)}_{\text{distribution at time } t} \underbrace{\varphi\Big(x_1 + \overset{\text{"of order } \sqrt{\delta t}\text{"}}{\overbrace{\mathring{\eta}}} + \delta t\, F(x_1,t)\Big)}_{\text{the expansion at small } \delta t \text{ is now easier}}$$



$\underbrace{}$

$\langle \mathring{\eta} \rangle = 0$, $\langle \mathring{\eta}^2 \rangle = 2D\,\delta t$

see (*)

$$\simeq \varphi(x_1) + \mathring{\eta}\,\varphi'(x_1) + \delta t\, F(x_1,t)\,\varphi'(x_1) + \frac{1}{2}\underbrace{\mathring{\eta}^2}_{} \varphi''(x_1) + O(\delta t^{3/2})$$

of order δt after averaging, see (*)

$$\langle \varphi(x) \rangle_{t+\delta t} = \int dx_1\, P(x_1,t) \Big\{ \varphi(x_1) + \delta t\, F(x_1,t)\varphi'(x_1) + D\,\delta t\, \varphi''(x_1) \Big\} + O(\delta t^{3/2})$$

$$\langle \varphi(x) \rangle_{t+\delta t} = \langle \varphi(x) \rangle_t + \delta t \big\langle F(x,t)\,\varphi'(x) + D\,\varphi''(x) \big\rangle_t + O(\delta t^{3/2})$$

One can now take the limit $\delta t \to 0$ by isolating $\dfrac{\langle \varphi(x)\rangle_{t+\delta t} - \langle \varphi(x)\rangle_t}{\delta t}$

which yields $$\boxed{\partial_t \langle \varphi(x)\rangle_t = \left\langle F(x,t)\varphi'(x) + D\varphi''(x)\right\rangle_t} \quad (*)$$

Remark: from the Langevin equation $\partial_t x = F(x,t) + \eta$

one is tempted to write $\partial_t \varphi(x_t) = \partial_t x_t \, \varphi'(x_t) = (F(x_t,t) + \eta_t)\varphi'(x_t)$

and to average as $$\partial_t \langle \varphi(x)\rangle_t = \langle \underbrace{F(x,t)\varphi'(x)}_{\text{"deterministic term"}} + \underbrace{\eta\, \varphi'(x)}_{\text{"stochastic term"}} \rangle$$

the "deterministic term" is the first term of $(*)$ and is easy to see.

the "stochastic term" $\langle \eta \varphi'(x)\rangle$ is not easy to determine and is equal to $D\langle\varphi''\rangle_t$ (see $(*)$).

→ this is due to the fact that $\eta(t)$ "white noise" is a very irregular function

Remark: $(*)$ is also true without averaging.

This is the Itō formula $$\boxed{\partial_t[\varphi(\overset{x(t)}{x},t)] = \partial_t\varphi(x,t) + \overset{\eta(t)\,\partial_x\varphi(x,t)}{+}\ F(x,t)\,\partial_x\varphi(x,t) + D\,\partial_x^2\varphi(x,t)}$$

Remark: time-derivative of averages: point of view of trajectories and distributions

From the point of view of trajectories: difference of values of the function φ on the same trajectory x_t. This difference is rather "singular" since x_t is "rough".

$$\langle \varphi(x_{t+\delta t}) - \varphi(x_t) \rangle = \underbrace{\int dx\, P(x,t+\delta t)\varphi(x)}_{} - \underbrace{\int dx\, P(x,t)\varphi(x)}_{}$$

Here x is a dummy variable and represents all possible values of x at time $t+\delta t$, of probability $P(x,t+\delta t)$

same as at time t

$$\langle \varphi(x_{t+\delta t}) - \varphi(x_t) \rangle = \int dx\, [P(x,t+\delta t) - P(x,t)]\,\varphi(x)$$

$$\Big\downarrow \ \delta t \to 0$$

$$\boxed{\underbrace{\partial_t \langle \varphi(x)\rangle_t}_{} = \underbrace{\int dx\, \partial_t P(x,t)\, \varphi(x)}_{}}$$

Viewpoint of trajectories (see computation p 10-11) or viewpoint of distributions, depending on what is useful.

Viewpoint of probability distributions

[Again, more simple to tackle as regards the regularity of functions & their expansion, since the prob. distribution $P(x,t)$ is "smooth".]

× Equation on the probability distribution :

$$\forall \varphi \qquad \partial_t \langle \varphi(x) \rangle_t = \langle F(x,t)\,\varphi'(x) + D\,\varphi''(x) \rangle_t$$

$$\int dx\, \partial_t P(x,t)\, \varphi(x) = \int dx \left\{ P(x,t)\, F(x,t)\, \varphi'(x) + D P(x,t)\, \varphi''(x) \right\}$$

$$\int dx\, \partial_t P(x,t)\, \varphi(x) = \int dx \left\{ -\partial_x \left[P(x,t) F(x,t) \right] + D\, \partial_x^2 P(x,t) \right\} \varphi(x)$$

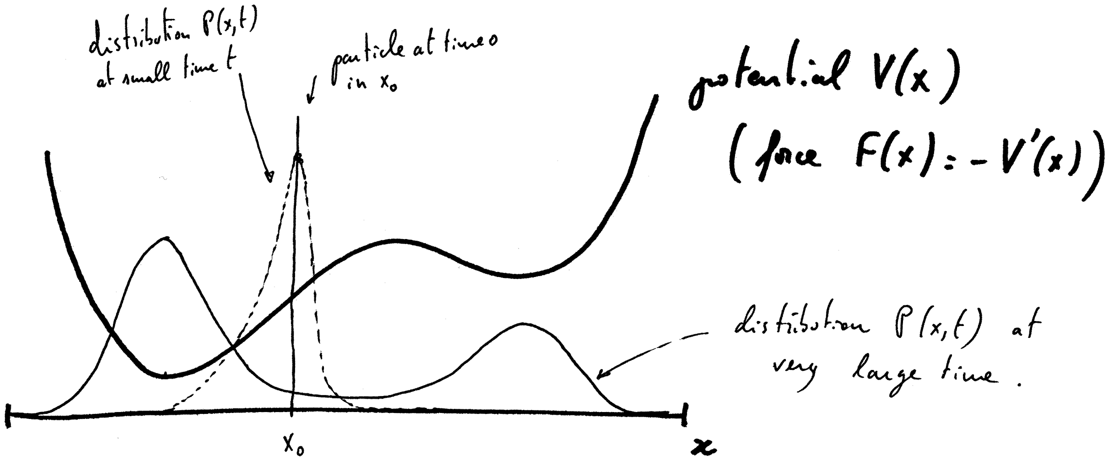
integrations by part, using $|\varphi| \xrightarrow[x\to\infty]{} 0$

This equation is valid for all functions $\varphi(x)$ decreasing fast enough at infinity

Hence:
$$\boxed{\partial_t P(x,t) = -\partial_x \left[F(x,t) P(x,t) \right] + D\, \partial_x^2 P(x,t)}$$

This is the Fokker-Planck equation for a 1D particle in a force $F(x,t)$.

× Example :



• Steady state: limit distribution $P_{st}(x) = \lim_{t\to\infty} P(x,t)$ if it exists

It verifies
$$0 = \partial_x \left[-F(x)\, P_{st}(x) + D\, \partial_x P_{st}(x) \right]$$
for a time-independent force $F(x,t)$

× Remark : for the free particle ($F=0$) one has $\partial_t P = D \partial_x^2 P$ (diffusion equation)

⚠ . on the real line : $P(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}} \xrightarrow[t\to\infty]{} 0$ there is no steady state

[long-time properties are catched here through scaling : $P(x,t) \sim \frac{1}{\sqrt{t}} \hat{P}(x/\sqrt{t})$]

. on an interval $[a,b]$: the steady state verifies
$$\begin{cases} \partial_x^2 P_{st}(x) = 0 \\ \partial_x P_{st}|_a = \partial_x P_{st}|_b = 0 \ : \text{no current on boundaries} \end{cases}$$

(for an isolated system)
$$\boxed{P_{st}(x) = \frac{1}{b-a}}$$
uniform distribution

Lecture on Complex Systems

- <u>Equilibrium steady state</u>: in a potential $V(x)$ at temperature T

× <u>remark</u>: if $P_{eq}(x) \propto e^{-\frac{1}{T}V(x)}$ (Boltzmann distribution)

one has $T\,\partial_x P_{eq}(x) = -V'(x)\,P_{eq}(x)$

hence: $\boxed{\text{The Boltzman distribution } P_{eq}(x) \propto e^{-\frac{1}{T}V(x)} \text{ is a steady-state solution of the Fokker-Planck equation } \partial_x \underbrace{\left[+V'(x)\,P_{eq}(x) + T\,\partial_x P_{eq}(x)\right]}_{=0} = 0}$

This provides a link between thermodynamics (Boltzman canonical ensemble)
stochastic dynamics (Langevin dynamics)
where $D = T$ (hence the interpretation of $\eta(t)$ as a <u>thermal noise</u>)

× <u>interpretation in terms of a probability current</u>

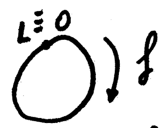
The FP equation writes $\partial_t P + \partial_x J[P] = 0$ with $J[P] = FP - D\partial_x P$

A steady-state distribution verifies $\partial_x J[P] = 0$ $\Rightarrow J[P] \equiv j$ (constant) [specific of 1D]
An equilibrium st.st. distribution — $J[P] = 0$ $j = 0$
when $j \neq 0$, we speak of a non-equilibrium steady state (NESS).

× <u>mean velocity in the system</u>: the example of $V = -fx$ i.e. $F(x) = f$

① $\langle \partial_t x \rangle = \langle f + \eta \rangle \Rightarrow \boxed{\langle \partial_t x \rangle \equiv v = f}$

② <u>for distribution and for the NESS</u>:
one considers periodic boundary conditions



$\langle \partial_t x \rangle = \int dx\, x\, \partial_t P(x,t) = -\int dx\, x\, \partial_x J[P]$ from Fokker-Planck

$\overset{\text{ipp}}{=} \int dx\, J[P]$ but $J[P] = j$ is a constant j

Hence: $\boxed{\text{the velocity is } v \equiv \langle \partial_t x \rangle = j = f}$

The system is in a true NESS ($j \neq 0$) iff $f \neq 0$.

It is not obvious to determine $P_{st}(x)$ for $f \neq 0$. See exercises.

↳ indeed a try as $P_{st}(x) \propto e^{-\frac{1}{T} f x}$ does <u>not</u> respect boundary conditions

- "*Equilibrium dynamics*" and *reversibility*:

  - if there exists a potential $V(x)$ such that $F(x) = -V'(x)$ (independent of time)
    and that $P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T}V(x)}$, $Z = \int dx\, e^{-\frac{1}{T}V(x)}$
    is the stationnary distribution $\underline{\underline{[J(P_{eq})=0]}}$

  then we say that the dynamics is an "equilibrium dynamics"
  and that $P_{eq}(x)$ is an "equilibrium steady state" (or "reversible measure").

  - otherwise, the dynamics is a "non-equilibrium dynamics". $\underline{\underline{[J(P_{st})\neq 0]}}$

→ There is a physical interpretation in terms of reversibility.
Let us fix the final time t and consider all trajectories $x(\tau)$ for $0 \leq \tau \leq t$.

- probability density of a trajectory $(x(\tau))_{0\leq\tau\leq t}$:

  simpliest way of obtaining it:

$$\mathrm{Prob}\,[x(\tau)] = \int \mathcal{D}\eta\ \delta\left(\partial_\tau x - F(x) - \eta\right) e^{-\frac{1}{4T}\int_0^t d\tau\ \eta(\tau)^2}$$

$$\boxed{\mathrm{Prob}\,[x(\tau)] = \exp\left\{-\frac{1}{4T}\int_0^t d\tau\ \left(\partial_\tau x - F(x)\right)^2\right\}}$$

This is justified e.g. from the expression of the average of an observable depending on $[x(\tau)]_{0\leq\tau\leq t}$:
$\langle \mathcal{O}[x(\tau)]\rangle = \int \mathcal{D}x\ \mathcal{O}[x(\tau)]\,\mathrm{Prob}[x]$ ...

- time-reversed trajectory: one defines $x^R(\tau) = x(t-\tau)$ $(0 \leq \tau \leq t)$
  which verifies $x^R(t) = x(0)$ $x^R(0) = x(t)$ $\partial_\tau x^R_{(t-\tau)} = -\partial_\tau x(\tau)$

- joint probability of history & initial condition in the equilibrium dynamics:

$$P_{eq}(x(0))\,\mathrm{Prob}\,[x(\tau)] \propto \underbrace{e^{-\frac{1}{T}V(x(0))}}_{= e^{-\frac{1}{T}V(x^R(t))}} \exp\left\{-\frac{1}{4T}\underbrace{\int_0^t d\tau\ \left(\partial_\tau x + V'(x)\right)^2}\right\}$$

$x \mapsto x^R$
$\tau \mapsto t-\tau$

so as to recover a +

$$\int_0^t d\tau\ \left(-\partial_\tau x^R + V'(x^R)\right)^2 = \int_0^t d\tau\left[\left(\partial_\tau x^R + V'(x^R)\right)^2 - 4\,\overbrace{\partial_\tau x^R\, V'(x^R)}^{\partial_\tau(V(x^R))}\right]$$

$$= -4\left(V(x^R(t)) - V(x^R(0))\right) + \int_0^t d\tau\left(\partial_\tau x^R + V'(x^R)\right)^2$$

those two terms cancel

this one remains

$$\propto e^{-\frac{1}{T}V(x^R(0))} \exp\left\{-\frac{1}{4T}\int_0^t d\tau\ \left(\partial_\tau x^R + V'(x^R)\right)^2\right\}$$

There is an equivalence btw
- $J[P] = 0$ (no prob. current)
- $P(x) \propto e^{-\frac{1}{T}V(x)}$
- reversibility in that sense

Hence:

$$P_{eq}(x(0))\,\mathrm{Prob}[x(\tau)] = P_{eq}(x^R(0))\,\mathrm{Prob}[x^R(\tau)]$$, i.e.

there is reversibility (trajectories and their time-reversed have the same probability, including initial eq. distrib.) when the dynamics is an "equilibrium dynamics".

otherwise not.

- Reversible dynamics, Doob transform and Schrödinger equation

Consider the "equilibrium dynamics" (also termed "reversible dyn.")

$D = T$ from now on

$$\partial_t x = -V'(x) + \eta \quad \Leftrightarrow \quad \partial_t P(x,t) = +\partial_x \left[V'(x) P(x,t) \right] + T \partial_x^2 P(x,t)$$

This equation for $P(x,t)$ is linear, first-order in time, and looks like Schrödinger's equation $-i\partial_t \Psi(x,t) = \left[\partial_x^2 - V_{quant}(x)\right] \Psi(x,t)$ on the wave function Ψ without however taking the same form.

Let's introduce $\boxed{P_{sym}(x,t) = e^{\frac{1}{2T}V(x)} P(x,t) = P_{eq}^{-1/2}(x)\, P(x,t)}$ (This is the Doob transform)

One easily checks that the Fokker-Planck equation is equivalent to:

$$\boxed{\begin{aligned} \partial_t P_{sym}(x,t) &= \left[T\partial_x^2 - V_{eff}(x)\right] P_{sym}(x,t) \\ V_{eff}(x,t) &= \frac{1}{4T}(V'(x))^2 - \frac{1}{2}V''(x) \end{aligned}}$$ effective potential

Hence: the FP equation for P_{sym} is equivalent to an [imaginary time] Schrödinger equation for P_{sym} in a potential $V_{eff}(x)$.

Again, this only works for "equilibrium dynamics".
Such a correspondence can be established in a much more general context, see later.

- Remark: correct form of the probability density of trajectories

The (quantum-like) Feynman path-integral would yield for the probability of histories:

$$P_{sym}(x,t) = \frac{1}{Z} e^{-\frac{1}{2T}V(x(0))} e^{-\frac{1}{4T}\int_0^t d\tau \left\{ \underbrace{(\partial_\tau x)^2 + (V'(x))^2}_{\text{up to time-boundary terms, corresponds to } (\partial_\tau x + V'(x))^2} \underbrace{- 2T V''(x)}_{\text{this term is absent from the expression p. 14}} \right\}}$$

→ This comes from a different choice of time-discretization of the Langevin equation (Itô vs Stratonovitch) → $e^{\int V''}$ can be seen as a Jacobian.

See A.W.C. Lau & T.C. Lubensky PRE 76 011123 (2007) for details

- Generalization to several interacting particles:

$x_i(t)$ each of them in a noise $\eta_i(t)$ $1 \leq i \leq N$

global interactions $F_i(\{x_j\}, t)$

$$\boxed{\begin{aligned} \partial_t x_i &= F_i(\{x_j\}, t) + \eta_i(t) \\ \langle \eta_i(t)\, \eta_j(t')\rangle &= 2\Delta_{ij}\, \delta(t-t') \end{aligned}}$$

with $\{\eta_j(t)\}$ white noises

Δ_{ij} invertible symmetric matrix

η_j's are uncorrelated if $\Delta_{ij} = D\,\delta_{ij}$

Gaussian distribution of the noise:

$$\mathrm{Prob}[\{\eta_j(t)\}] \propto \exp\left(-\frac{1}{2}\int_0^t d\tau \sum_{ij} \eta_i(\tau)\left((2\Delta)^{-1}\right)_{ij}\eta_j(\tau)\right)$$

if $\Delta_{ij} = \delta_{ij}$

$$\propto \exp\left(-\frac{1}{4D}\int_0^t d\tau \sum_i \eta_i(\tau)^2\right)$$

× Fokker - Planck equation: notation $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$

$$\boxed{\partial_t P(\overset{x}{\overbrace{\{x_j\}}}, t) = \underbrace{-\sum_i \partial_i\left(F_i(x,t)\, P(x,t)\right)}_{\text{deterministic contribution}} + \underbrace{\sum_{ij} \Delta_{ij}\, \partial_i \partial_j P(x,t)}_{\text{(correlated) diffusive contribut}^\circ}}$$

in matrix notations:

$$\partial_t P = -\vec{\nabla}\cdot\left(\vec{F}(x,t)\, P(x,t)\right) + \vec{\nabla}\cdot\left(\overset{\text{matrix } \Delta_{ij}}{\Delta}\, \vec{\nabla} P(x,t)\right)$$

if the noise is uncorrelated $(\Delta_{ij} = D\delta_{ij})$

$$\partial_t P = -\vec{\nabla}\cdot\left(\vec{F}(x,t)\, P(x,t)\right) + D\,\vec{\nabla}^2 P(x,t)$$

× Itô formula:

$$\boxed{\partial_t \varphi(x,t) = \sum_i F_i(x,t)\, \partial_i \varphi(x,t) + \sum_{ij} \Delta_{ij}\, \partial_i \partial_j \varphi(x,t)}$$

$$\boxed{+ \sum_i \eta_i(x,t)\, \partial_i \varphi(x,t)}$$

× Equilibrium :

When $\forall i,\ \underbrace{F_i(x,t)}_{\text{forces acting on particle } i} = -\partial_i \underbrace{V(x)}_{\text{common potential to all particles}}$

describing interactions between particles and also with the environment

i.e. when "forces are conservative"
i.e. when "forces derive from a potential"
then the Boltzmann-Gibbs measure is an equilibrium steady state

① For non-correlated noises $\langle \eta_i \eta_j \rangle = 2D\, \delta_{ij}\, \delta(t-t')$; denoting $\boxed{D=T}$

$$\boxed{P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T}V(x)}} \qquad \text{with } Z = \int dx\ e^{-\frac{1}{T}V(x)}$$

verifies $\partial_i P_{eq}(x) = -\frac{1}{T}\partial_i V(x)\ P_{eq}(x)$ hence $\partial_i V\, P_{eq} + T P_{eq} = 0$

but the FP equation writes $\partial_t P = +\sum_i \partial_i \left[\underbrace{\partial_i V P + T P}\right]$

hence $\underline{P_{eq}(x)}$ is an equilibrium distribution — all components of the current are 0

② For correlated noises $\langle \eta_i(t)\eta_j(t')\rangle = 2\Delta_{ij}\, \delta(t-t')$:

$$P_{eq}(x) = \frac{1}{Z} \exp\left(-W(x)\right) \Rightarrow \partial_i P_{eq} = -\partial_i W\ P_{eq}$$

searching for a solution of $\partial_i V P_{eq} + \sum_j \Delta_{ij} \partial_j P_{eq} = 0$ is equivalent to

$$\forall i,\ \partial_i V = \sum_j \Delta_{ij} \partial_j W \quad \text{ie} \quad \forall i,\ \boxed{\partial_j W = \sum_i (\Delta^{-1})_{ij} \partial_j V}$$

A necessary and sufficient condition for this equation to have a solution (using Poincaré's lemma) is : $\forall i,j,\ \sum_k (\Delta^{-1})_{ki} \partial_j V = \sum_k (\Delta^{-1})_{kj} \partial_i V$

A possible solution is for quadratic potentials : $V = \sum_{ij} \Delta_{ij} x_i x_j$

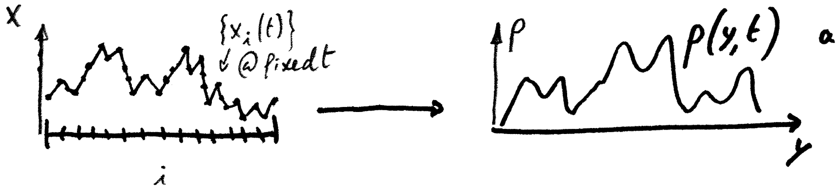
But otherwise there is no generic characterisation of equilibrium with correlated noise.

- Generalisation to several coupled noises: continuum

$x \equiv \rho$
$y \equiv i$

$$x_i(t) \longrightarrow \rho(y,t)$$

$$\eta_i(t) \longrightarrow \eta(y,t)$$



at fixed t

one will call $\rho(y,t)$ a "profile", but the description holds in generality

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\Delta_{ij}\,\delta(t-t') \longrightarrow \langle \eta(y,t)\,\eta(y',t')\rangle = \underbrace{R(y'-y)}_{\text{describes correlations in direction } y}\,\delta(t-t')$$

$\left[R(y) = 2D\,\delta(y'-y) \text{ for uncorrelated noise}\right]$

$$P(x,t) \longrightarrow P[\rho,t]$$ (functional) probability density of the profile $\rho(y)$, at time t

- Langevin equation: from $\partial_t x_i = F_i(x,t) + \eta_i$ (see p. 1.16) one goes to

$$\boxed{\partial_t \rho(y,t) = \mathcal{F}[\rho;y,t] + \eta(y,t)}$$

- Fokker-Planck equation: from $\partial_t P(x,t) = -\sum_i \partial_i\big(F_i(x,t)P(x,t)\big) + \sum_{ij}\Delta_{ij}\,\partial_i\partial_j P(x,t)$ one has

$$\boxed{\partial_t P[\rho,t] = -\int dy\,\frac{\delta}{\delta\rho(y)}\Big(\mathcal{F}[\rho;y,t]\,P[\rho,y]\Big) + \frac{1}{2}\int dy\,dy'\,R(y'-y)\,\frac{\delta^2 P[\rho,t)}{\delta\rho(y)\,\delta\rho(y')}}$$

- Itō formula:

$$\partial_t\big(\varphi[\rho,t]\big) = \partial_t\varphi[\rho,t] + \int dy\,\mathcal{F}[\rho;y,t]\,\frac{\delta\varphi[\rho,t]}{\delta\rho(y)} + \frac{1}{2}\int dy\,dy'\,R(y'-y)\,\frac{\delta^2\varphi[\rho,t]}{\delta\rho(y)\,\delta\rho(y')}$$

($\rho = \rho(y,t)$)

- Example of "equilibrium (i.e. reversible) dynamics": $\mathcal{F}[\rho;y,t] = \dfrac{\delta\mathcal{H}[\rho]}{\delta\rho(y)}$

Then for an uncorrelated noise $R(y) = 2T\,\delta(y)$

$\boxed{P_{eq}[\rho] \propto \exp\left(-\frac{1}{T}\mathcal{H}[\rho]\right)}$ is a solution of the FP equation, with 0 current:

$$-\mathcal{F}[\rho;y,t]\,P_{eq}[\rho] + \frac{1}{2T}\,\frac{\delta P_{eq}[\rho]}{\delta\rho(y)} = 0$$

- Generalization to position-dependent noise:

Bad writing:

$$\boxed{\partial_t x = F(x,t) + \sqrt{2D(x,t)}\ \eta(t)} \qquad \text{with } \langle \eta(t)\eta(t')\rangle = \delta(t-t')$$

$D(x,t) > 0$

Fokker-Planck: $\quad \mathrm{Prob}[x(t)] \propto \exp\left(-\frac{1}{2}\int_0^t d\tau\, \frac{(\partial_\tau x - F(x,t))^2}{2D(x,t)}\right)$

$$\boxed{\partial_t P(x,t) = -\partial_x\big(F(x,t)P(x,t)\big) + \partial_x^2\big(D(x,t)P(x,t)\big)}$$

- With several coupled noises:

$$\boxed{\partial_t x_i = F_i(x,t) + \sqrt{2D(x,t)}\ \eta_i(t)} \qquad \langle \eta_i(t)\eta_j(t')\rangle = \Delta_{ij}\delta(t-t')$$

$$\boxed{\partial_t P(x,t) = -\sum_i \partial_i\big(F_i(x,t)P(x,t)\big) + \sum_{ij}\partial_i\partial_j\big(\Delta_{ij}\,D(x,t)P(x,t)\big)}$$

- Generalization to fields: $\quad x_i(t) \mapsto \rho(y,t)$

$$\partial_t \rho(y,t) = \mathcal{F}\big(\rho(x,t),t\big) + \sqrt{2D(\rho(y,t))}\ \eta(y,t) \qquad \langle \eta(y,t)\eta(y',t')\rangle = R(y-y')\delta(t-t')$$

$$\partial_t P[\rho(y),t] = -\int dy\, \frac{\delta}{\delta\rho(y)}\Big(F(\rho(y),t)P[\rho(y),t]\Big) + \int dy\,dy'\,\frac{\delta^2}{\delta\rho(y)\delta\rho(y')}\Big[R(y-y')P[\rho,t] \times D(\rho(y))\Big]$$

Remark : Backward Fokker-Planck equation:

We have seen that $P(x,t|x_0 t_0) = P(x,t)$ verifies the forward FP equation

$$\partial_t P(x,t) = -\partial_x \left(F(x,t)P(x,t)\right) + D\partial_x^2 P(x,t)$$

We might be interested in the derivatives $\partial_{x_0}, \partial_{t_0}$ w.r.t. the initial conditions (e.g. to tackle 1^{st} passage problems).

As on page 1.10, we integrate the Langevin equation $\dot{x} = +F(x,t) + \eta$ $\quad \delta t > 0$

between $t_0 - \delta t$ and t_0 to write $\underbrace{x(t_0)}_{\equiv x_0} = \underbrace{x(t_0-\delta t)}_{\equiv x_{-1}} + \delta t\, F(x_{-1}, t-\delta t) + \underbrace{\eta^0}_{\int_{t_0-\delta t}^{t_0} d\tau\, \eta(\tau)}$

with η^0 Gaussian, $\langle \eta^0 \rangle = 0$ $\quad \langle \eta^{0\,2} \rangle = 2D\delta t$

$$\Downarrow$$

$$x_{-1} = x_0 - \delta t\, F(x_0,t) - \eta^0$$

Then, for any function $f(x)$ one has:

$$\int dx\, f(x,t) P(x,t|x_0,t_0) = \int dx\, f(x,t) \int d\eta^0\, P(\eta^0)\, \underbrace{P(x,t\,|\, x_0 - \eta^0 - \delta t\, \overbrace{F(x_0,t)}, t_0 - \delta t)}$$

or $F(x_{-1}, t-\delta t)$ but this is the same at the order δt we are interested in

we perform the average $\int d\eta^0 P(\eta^0)$ through $\langle \eta^0 \rangle = 0$, $\langle \eta^{0\,2} \rangle = 2D\delta t$

$$\simeq P(x,t|x_0,t_0-\delta t) - \eta^0 \partial_{x_0} P(x,t|x_0,t_0) - \delta t\, \underset{F(x_0,t)}{} \partial_{x_0} P(x,t|x_0,t_0) - \frac{1}{2}\eta^{0\,2} \partial_{x_0}^2 P(x,t|x_0,t_0) + \mathcal{O}(\delta t^{3/2})$$

one has to expand up to order δt

Collecting:

$$\int dx\, f(x,t) \overbrace{\frac{P(x,t|x_0,t_0) - P(x,t|x_0,t_0-\delta t)}{\delta t}}^{\xrightarrow{\delta t\to 0}\ \partial_{t_0} P(x,t|x_0,t_0)} = \int dx\, f(x,t)\left[-F(x_0,t)\partial_{x_0} P(x,t|x_0,t_0) - D\partial_{x_0}^2 P(x,t|x_0,t_0)\right]$$

And finally:

Note the change of sign & the position of ∂_{x_0}

$$\boxed{-\partial_{t_0} P(x,t|x_0,t_0) = F(x_0,t)\,\partial_{x_0} P(x,t|x_0,t_0) + D\partial_{x_0}^2 P(x,t|x_0,t_0)}$$

This is the backward Fokker-Planck equation, which describes the variations of the probability density with respect to its initial conditions. $\quad P(x,t|x_0t_0) = e^{(t-t_0)\mathbb{W}} \underset{\text{indep. of time}}{P_0(x_0)}$

For homogeneous processes where $P(x,t|x_0 t_0)$ is a function of $t-t_0$ only, e.g. if $F(x,t) = F(x)$ one has $\partial_{t_0} = -\partial_t$ and hence

$$\boxed{\partial_t P(x,t|x_0,t_0) = F(x_0\ )\,\partial_{x_0} P(x,t|x_0,t_0) + D\partial_{x_0}^2 P(x,t|x_0,t_0)}$$