

Equation of evolution for the probability density $P(x, t)$:

Lecture on
Complex Systems

$$\frac{\partial x(t)}{\partial t} = \underbrace{F(x(t), t)}_{\text{"deterministic force"}} + \underbrace{\eta(t)}_{\text{"stochastic force"}}$$

$F(x, t)$ is assumed to be regular enough

Remark: without forces ($F=0$) this was denoted $x(t)$ previously; now that $x(t)$ evolves with $F(x, t)$, we call this $B(t)$

where we denote by definition

$$\overset{o}{\eta_t} = \int_t^{t+\delta t} \eta(\tau) d\tau = B(t+\delta t) - B(t)$$

$B(t) = \int_0^t \eta(\tau) d\tau$: Brownian motion

$$\text{we have seen that } \langle \overset{o}{\eta_t} \overset{o}{\eta_t} \rangle = \langle (B(t+\delta t) - B(t))^2 \rangle = C(\delta t) = \underline{\underline{2D\delta t}} \quad (\star)$$

↪ 1) when expanding, $\overset{o}{\eta_t}$ is of order $\sqrt{\delta t}$.

$$x_{t+\delta t} - x_t = \delta t F(x_t, t) + \overset{o}{\eta_t}$$

Remark: one could take for $\overset{o}{\eta_t}$: $\begin{cases} +\delta x & \text{prob } \frac{1}{2} \\ -\delta x & \frac{1}{2} \\ 0 & 1-2 \end{cases}$

As for the free case ($F=0$) one could take the limit $\underset{\delta t \rightarrow 0}{\lim} \frac{\delta x}{\delta t}$ fixed

× 1st try: directly expand $P(x, t+\delta t)$

$$P(x, t+\delta t) = \int dx_1 P(x_1, t) P(x, t+\delta t | x_1, t)$$

$$P(x, t+\delta t) - P(x, t) = \int dx_1 P(x_1, t) [P(x, t+\delta t | x_1, t) - \delta(x - x_1)]$$

a small δt expansion would be difficult
(it is difficult to expand around a Dirac delta)
→ in such situations, integrate over a "test function":

× 2nd try: consider the time-evolution of the average of an observable $\varphi(x)$:

Consider an observable $\varphi(x)$, vanishing at $|x| \rightarrow \infty$ (a "test function", for

Average value
at time $t+\delta t$

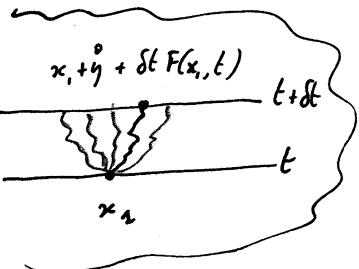
"of order $\sqrt{\delta t}$ " mathematicians)

$$\langle \varphi(x) \rangle_{t+\delta t} = \int dx_1 \underbrace{\int d\overset{o}{\eta} P[\overset{o}{\eta}]}_{\substack{\text{possible positions at time } t \\ \text{average over the noise } \overset{o}{\eta}}} \underbrace{P(x_1, t)}_{\substack{\text{distribution} \\ \text{at time } t}} \underbrace{\varphi(x_1 + \overset{o}{\eta} + \delta t F(x_1, t))}_{\substack{\text{"of order } \sqrt{\delta t}" \\ \text{the expansion at small } \delta t \text{ is now easier}}}$$

$$\approx \varphi(x_1) + \overset{o}{\eta} \varphi'(x_1) + \delta t F(x_1, t) \varphi'(x_1)$$

$$+ \frac{1}{2} \overset{o}{\eta}^2 \varphi''(x_1) + \mathcal{O}(\delta t^{3/2})$$

of order δt after averaging, see (\star)



$$\langle \varphi(x) \rangle_{t+\delta t} = \int dx_1 P(x_1, t) \{ \varphi(x_1) + \delta t F(x_1, t) \varphi'(x_1) + D \delta t \varphi''(x_1) \} + \mathcal{O}(\delta t^{3/2})$$

$$\langle \varphi(x) \rangle_{t+\delta t} = \langle \varphi(x) \rangle_t + \delta t \langle F(x, t) \varphi'(x) + D \varphi''(x) \rangle_t + \mathcal{O}(\delta t^{3/2})$$

One can now take the limit $\delta t \rightarrow 0$ by isolating $\frac{\langle \varphi(x) \rangle_{t+\delta t} - \langle \varphi(x) \rangle_t}{\delta t}$

which yields

$$\partial_t \langle \varphi(x) \rangle_t = \langle F(x, t) \varphi'(x) + D \varphi''(x) \rangle_t$$

(1.11)
lecture on
Complex System

Remark: from the Langevin equation $\partial_t x = F(x, t) + \eta$

one is tempted to write $\partial_t \varphi(x_t) = \partial_t x_t \varphi'(x_t) = (F(x_t, t) + \eta_t) \varphi'(x_t)$

and to average as $\partial_t \langle \varphi(x) \rangle_t = \underbrace{\langle F(x, t) \varphi'(x) \rangle}_{\text{"deterministic term"}} + \underbrace{\langle \eta \varphi'(x) \rangle}_{\text{"stochastic term"}}$

the "deterministic term" is the first term of ④ and is easy to see.

the "stochastic term" $\langle \eta \varphi'(x) \rangle$ is not easy to determine and is equal to $D \langle \varphi''(x) \rangle$,
"white noise" (see ①).

→ this is due to the fact that $\eta(t)$ is a very irregular function

Remark: ④ is also true without averaging.

This is the Ito formula $\partial_t [\varphi(x, t)] = \partial_t \varphi(x, t) + \underbrace{F(x, t) \partial_x \varphi(x, t)}_{\eta(t)} + D \partial_x^2 \varphi(x, t)$

Remark: time-derivative of averages: points of view of trajectories and distributions

From the point of view of trajectories: difference of values of the function φ on the same trajectory x_t . This difference is rather "singular" since x_t is "rough".

$$\langle \varphi(x_{t+\delta t}) - \varphi(x_t) \rangle = \underbrace{\int dx P(x, t+\delta t) \varphi(x)} - \underbrace{\int dx P(x, t) \varphi(x)}$$

Here x is a dummy variable and represents all possible values of x at time $t+\delta t$, of probab. l.f. $P(x, t+\delta t)$

← same as
at time t

$$\langle \varphi(x_{t+\delta t}) - \varphi(x_t) \rangle = \int dx [P(x, t+\delta t) - P(x, t)] \varphi(x)$$

$$\downarrow \delta t \rightarrow 0$$

$$\underbrace{\partial_t \langle \varphi(x) \rangle_t}_{\text{Viewpoint of trajectories}} = \underbrace{\int dx \partial_t P(x, t) \varphi(x)}_{\text{Viewpoint of probability distributions}}$$

(see computation p 10-11) or
viewpoint of distributions., depending
on what is useful.

Viewpoint of probability distributions

[Again, more simple to tackle as regards the regularity of functions & their expansion, since the prob. distribution $P(x, t)$ is "smooth".]

x Equation on the probability distribution :

Lecture on
Complex Systems (1.12)

$$\text{H}\varphi \quad \partial_t \langle \varphi(x) \rangle_t = \langle F(x,t) \varphi'(x) + D \varphi''(x) \rangle_t$$

$$\int dx \partial_t P(x,t) \varphi(x) = \int dx \left\{ P(x,t) F(x,t) \varphi'(x) + D P(x,t) \varphi''(x) \right\}$$

$$\int dx \partial_t P(x,t) \varphi(x) = \int dx \left\{ -\partial_x [P(x,t) F(x,t)] + D \partial_x^2 P(x,t) \right\} \varphi(x)$$

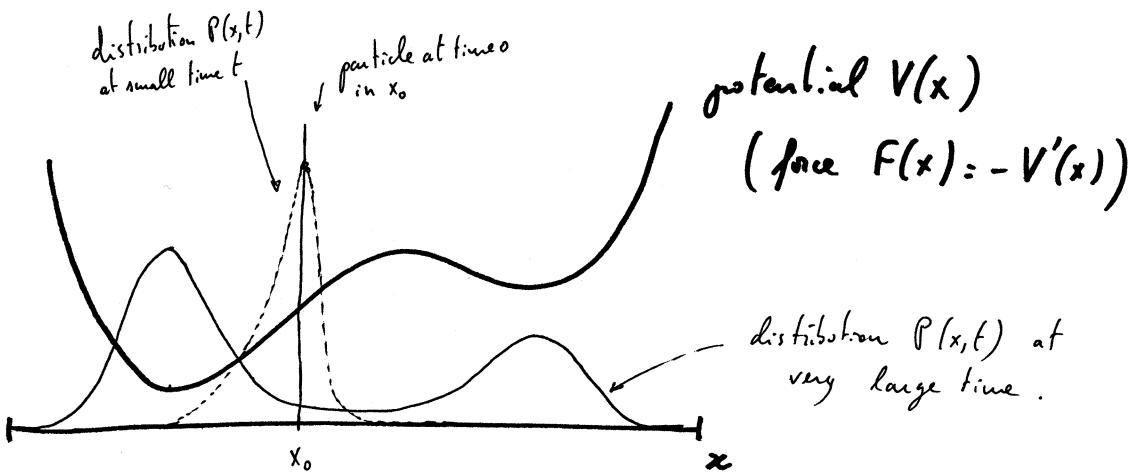
) integration by part, using $\lim_{x \rightarrow \infty} \varphi(x) = 0$

This equation is valid for all functions $\varphi(x)$ decreasing fast enough at infinity.

Hence: $\partial_t P(x,t) = -\partial_x [F(x,t) P(x,t)] + D \partial_x^2 P(x,t)$

This is the Fokker-Planck equation for a 1D particle in a force $F(x,t)$.

x Example :



• Steady state: limit distribution $P_{st}(x) = \lim_{t \rightarrow \infty} P(x,t)$ if it exists

It verifies $0 = \partial_x [-F(x) P_{st}(x) + D \partial_x P_{st}(x)]$ for a time-independent force $F(x,t)$

x Remark : for the free particle ($F=0$) one has $\partial_t P = D \partial_x^2 P$ (diffusion equation)

⚠ . on the real line : $P(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$ $\xrightarrow[t \rightarrow \infty]{} 0$ there is no steady state

[long-time properties are caught here through scaling: $P(x,t) \sim \frac{1}{\sqrt{t}} \tilde{P}(x/\sqrt{t})$]

. on an interval $[a,b]$: the steady state verifies $\begin{cases} \partial_x^2 P_{st}(x) = 0 \\ \partial_x P_{st}|_a = \partial_x P_{st}|_b = 0 \end{cases}$: no current on boundaries

(for an isolated system)

$$P_{st}(x) = \frac{1}{b-a}$$

uniform distribution

- Equilibrium steady state: in a potential $V(x)$
at temperature T

* remark: if $P(x) \propto e^{-\frac{1}{T}V(x)}$ (Boltzmann distribution)

$$\text{one has } T \partial_x P_{eq}(x) = -V'(x) P_{eq}(x)$$

hence:

The Boltzmann distribution $P_{eq}(x) \propto e^{-\frac{1}{T}V(x)}$ is a steady-state solution of the Fokker-Planck equation $\partial_x [+ V'(x) P_{eq}(x) + T \partial_x P_{eq}(x)] = 0$

This provides a link between thermodynamics (Boltzmann canonical ensemble)
stochastic dynamics (Langevin dynamics)
where $D = T$ (hence the interpretation of $\gamma(t)$ as a thermal noise)

* interpretation in terms of a probability current

The FP equation writes $\partial_t P + \partial_x J(P) = 0$ with $J(P) = F P - D \partial_x P$

A steady-state distribution verifies $\partial_x J(P) = 0 \Rightarrow J(P) = j$ (constant)

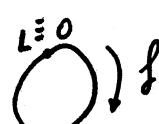
An equilibrium st.st. distribution — $J(P) = 0$ $j = 0$ [precise of 1D]
when $j \neq 0$, we speak of a non-equilibrium steady state (NESS).

* mean velocity in the system: the example of $V = -fx$ i.e. $F(x) = f$

$$\textcircled{1} \quad \langle \partial_t x \rangle = \langle f + \gamma \rangle \Rightarrow \boxed{\langle \partial_t x \rangle = v = f}$$

\textcircled{2} for distribution and for the NESS:

one considers periodic boundary conditions



$$\langle \partial_t x \rangle = \int dx x \partial_t P(x,t) = - \int dx x \partial_x J(P) \quad \text{from Fokker-Planck}$$

$$\text{iff } \int dx J(P) \quad \text{but } J(P) = j \text{ is a constant } j$$

Hence: $\boxed{\text{the velocity is } v = \langle \partial_t x \rangle = j = f}$

The system is in a true NESS ($j \neq 0$) iff $f \neq 0$.

It is not obvious to determine $P_{st}(x)$ for $f \neq 0$. See exercises.

↳ indeed a try as $P_{st}(x) \propto e^{-\frac{1}{T}f x}$ does not respect boundary conditions

- "Equilibrium dynamics" and reversibility: independent of time
 - if there exists a potential $V(x)$ such that $F(x) = -V'(x)$ and that $P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T} V(x)}$, $Z = \int dx e^{-\frac{1}{T} V(x)}$ is the stationary distribution $[J(P) = 0]$
 - then we say that the dynamics is an "equilibrium dynamics" and that $P_{eq}(x)$ is an "equilibrium steady state" (or "reversible measure").
 - otherwise, the dynamics is a "non-equilibrium dynamics". $[J(P_{st}) \neq 0]$

→ There is a physical interpretation in terms of reversibility.

let us fix the final time t and consider all trajectories $x(\tau)$ for $0 \leq \tau \leq t$.

- * probability density of a trajectory $(x(\tau))_{0 \leq \tau \leq t}$:

simpliest way of obtaining it:

$$\text{Prob}[x(\tau)] = \int \mathcal{D}\eta \delta(\partial_\tau x - F(x) - \eta) e^{-\frac{1}{4T} \int_0^t d\tau \eta(\tau)^2}$$

$$\boxed{\text{Prob}[x(\tau)] = \exp \left\{ -\frac{1}{4T} \int_0^t d\tau (\partial_\tau x - F(x))^2 \right\}}$$

This is justified e.g. from the expression of the average of an observable depending on $[x(\tau)]_{0 \leq \tau \leq t}$:

$$\langle O[x(\tau)] \rangle = \int \mathcal{D}x O[x(\tau)] \text{Prob}[x]$$

- * time-reversed trajectory: one defines $x^R(\tau) = x(t-\tau)$ ($0 \leq \tau \leq t$) which verifies $x^R(t) = x(0)$ $x^R(0) = x(t)$ $\partial_\tau x^R(t-\tau) = -\partial_\tau x(\tau)$
- * joint probability of history & initial condition in the equilibrium dynamics:

$$P_{eq}(x(0)) \text{Prob}[x(\tau)] \propto e^{-\frac{1}{T} V(x(0))} \exp \left\{ -\frac{1}{4T} \int_0^t d\tau (\partial_\tau x + V'(x))^2 \right\}$$

to auto-recover a +

\downarrow $\partial_\tau (V(x^R))$

$$\begin{matrix} x \mapsto x^R \\ \tau \mapsto t-\tau \end{matrix} \quad \curvearrowleft \quad = e^{-\frac{1}{T} V(x^R(t))}$$

$$\int_0^t d\tau (-\partial_\tau x^R + V'(x^R))^2 = \int_0^t \left[(\partial_\tau x^R + V'(x^R))^2 - 4 \partial_\tau x^R V'(x^R) \right]$$

There is an equivalence btw
 • $J(P) = 0$ (no prob. current)
 • $P(x) \propto e^{-\frac{1}{T} V(x)}$
 • reversibility in that sense

$$\uparrow \quad \text{those two terms cancel}$$

$$= -4(V(x^R(t)) - V(x^R(0))) + \int_0^t (\partial_\tau x^R + V'(x^R))^2$$

$$\downarrow \quad \text{this one remains}$$

$$\propto e^{-\frac{1}{T} V(x^R(0))} \exp \left\{ -\frac{1}{4T} \int_0^t d\tau (\partial_\tau x^R + V'(x^R))^2 \right\}$$

Hence:

$$P_{eq}(x(0)) \text{Prob}[x(\tau)] = P_{eq}(x^R(0)) \text{Prob}[x^R(\tau)], \text{i.e.}$$

otherwise not.

there is reversibility (trajectories and their time-reversed have the same probability, including initial eq. distrib.) when the dynamics is an "equilibrium dynamics".

Reversible dynamics, Doob transform and Schrödinger equation

Lecture on (6.15)

Consider the "equilibrium dynamics" (also termed "reversible dyn.")

Complex Systems
D = T from now on

$$\partial_t x = -V'(x) + \eta \quad \Leftrightarrow \quad \partial_t P(x,t) = +\partial_x [V'(x) P(x,t)] + T \partial_x^2 P(x,t)$$

This equation for $P(x,t)$ is linear, first-order in time, and looks like Schrödinger's equation $-i\partial_t \Psi(x,t) = [\partial_x^2 - V_{\text{quant}}(x)] \Psi(x,t)$ on the wave function Ψ without however taking the same form.

let's introduce $P_{\text{sym}}(x,t) = e^{\frac{i}{2T} V(x)} P(x,t) = P_{\text{eq}}^{-1/2}(x) P(x,t)$ (This is the Doob transform)

One easily checks that the Fokker-Planck equation is equivalent to:

$$\begin{aligned} \partial_t P_{\text{sym}}(x,t) &= [T \partial_x^2 - V_{\text{eff}}(x)] P_{\text{sym}}(x,t) \\ V_{\text{eff}}(x,t) &= \frac{1}{4T} (V'(x))^2 - \frac{1}{2} V''(x) \end{aligned}$$

effective potential

Hence: { the FP equation for P_{sym} is equivalent to an [imaginary time] Schrödinger equation for P_{sym} in a potential $V_{\text{eff}}(x)$. }

Again, this only works for "equilibrium dynamics".

Such a correspondence can be established in a much more general context, see later.

Remark: correct form of the probability density of trajectories

The (quantum-like) Feynman path-integral would yield for the probability of histories:

$$P_{\text{sym}}(x,t) = \frac{1}{2} e^{-\frac{1}{2T} V(x_{(0)})} e^{-\frac{1}{4T} \int_0^t \underbrace{\left\{ (\partial_\tau x)^2 + (V'(x))^2 \right\}}_{\text{up to time-boundary terms, corresponds to } (\partial_\tau x + V'(x))^2} - 2T V''(x) \}$$

this term is absent from the expression p. 14

→ This comes from a different choice of time-discretization of the Langevin equation (Itô vs Stratonovitch) → $e^{\int V' d\tau}$ can be seen as a Jacobian.

See A.W.C. Lau & T.C. Lubensky PRE 76 011123 (2007) for details

• Generalization to several interacting particles:

$x_i(t)$ each of them in a noise $\eta_i(t)$ $1 \leq i \leq N$
global interactions $F_i(\{x_j\}, t)$

$$\partial_t x_i = F_i(\{x_j\}, t) + \eta_i(t)$$

$$\langle \eta_i(t) \eta_j(t') \rangle = 2 \Delta_{ij} \delta(t-t')$$

with $\{\eta_j(t)\}$ white noises

Δ_{ij} invertible asymmetric matrix

η_j 's are uncorrelated if $\Delta_{ij} = D \delta_{ij}$

Gaussian distribution of the noise:

$$\text{Prob}[\{\eta_i(t)\}] = \exp \left(-\frac{1}{2} \int_0^t d\tau \sum_{ij} \eta_i(\tau) ((2D)^{-1})_{ij} \eta_j(\tau) \right)$$

$$\text{if } \Delta_{ij} = \delta_{ij} \quad \approx \exp \left(-\frac{1}{4D} \int_0^t d\tau \sum_i \eta_i(\tau)^2 \right)$$

• Fokker-Planck equation: notation $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$

$$\partial_t P(\{x_j\}, t) = - \underbrace{\sum_i \partial_i (F_i(x, t) P(x, t))}_{\text{deterministic contribution}} + \underbrace{\sum_{ij} \Delta_{ij} \partial_i \partial_j P(x, t)}_{(\text{unrelated}) \text{ diffusive contrib.}}$$

in matrix notations:

$$\partial_t P = - \vec{\nabla} \cdot (\vec{F}(x, t) P(x, t)) + \vec{\nabla} \cdot (\Delta^{-1} \vec{\nabla} P(x, t))$$

if the noise is uncorrelated ($\Delta_{ij} = D \delta_{ij}$)

$$\partial_t P = - \vec{\nabla} \cdot (\vec{F}(x, t) P(x, t)) + D \vec{\nabla}^2 P(x, t)$$

• Ito formula:

$$\begin{aligned} \partial_t \varphi(x, t) &= \sum_i F_i(x, t) \partial_i \varphi(x, t) + \sum_{ij} \Delta_{ij} \partial_i \partial_j \varphi(x, t) \\ &\quad + \sum_i \eta_i(x, t) \partial_i \varphi(x, t) \end{aligned}$$

* Equi-librium :

$$\text{When } V_i, F_i(x, t) = -\partial_i V(x)$$

force acting on
particle i

common potential to all particles
describing interactions between particles
and also with the environment

i.e. when "forces are conservative"

i.e. when "forces derive from a potential"

then the Boltzmann-Gibbs measure is an equilibrium steady state

① For non-correlated noises $\langle \eta_i(t) \eta_j(t') \rangle = 2 D_{ij} \delta(t-t')$; denoting $D=T$

$$P_{eq}(x) = \frac{1}{Z} e^{-\frac{1}{T} V(x)} \quad \text{with } Z = \int dx e^{-\frac{1}{T} V(x)}$$

verifies $\partial_i P_{eq}(x) = -\frac{1}{T} \partial_i V(x) P_{eq}(x)$ hence $\underbrace{\partial_i V P_{eq}}_{=0} + T P_{eq} = 0$
but the FP equation writes $\partial_t P = + \sum_i \partial_i \left[\underbrace{\partial_i V P}_{=0} + T P \right]$

hence $P_{eq}(x)$ is an equilibrium distribution all components of the current are 0

② For unrelated noises $\langle \eta_i(t) \eta_j(t') \rangle = 2 \Delta_{ij} \delta(t-t')$:

$$P_{eq}(x) = \frac{1}{Z} \exp(-W(x)) \Rightarrow \partial_i P_{eq} = -\partial_i W P_{eq}$$

searching for a solution of $\partial_i V P_{eq} + \sum_j \Delta_{ij} \partial_j P_{eq} = 0$ is equivalent to

$$\forall i, \partial_i V = \sum_j \Delta_{ij} \partial_j W \quad \text{i.e. } \forall i, \partial_j W = \sum_j (\Delta^i)_{ij} \partial_j V$$

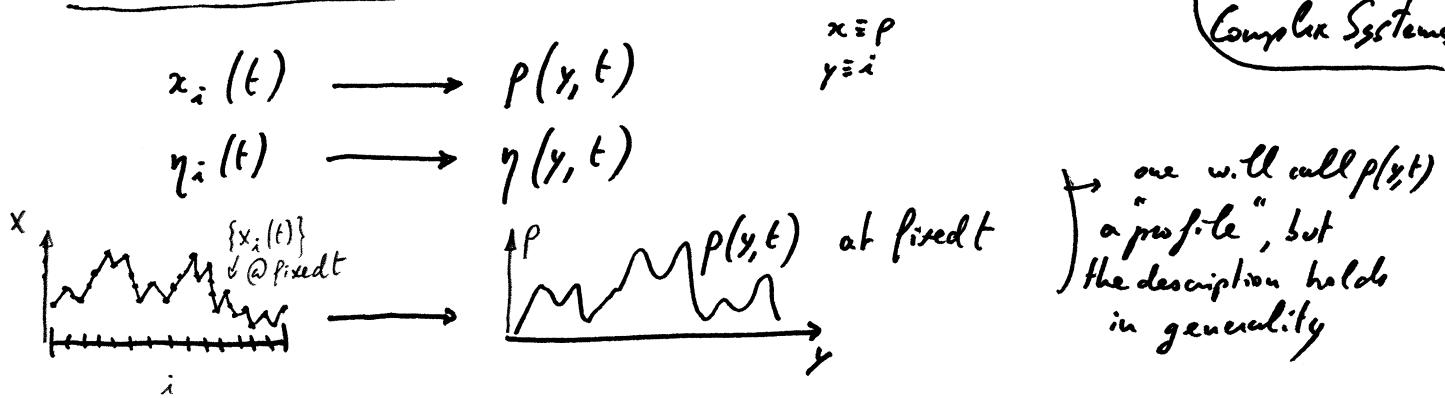
A necessary and sufficient condition for this equation to have a solution (using Poincaré's lemma) is: $\forall i, j, \sum_k (\Delta^i)_{ki} \partial_k V = \sum_k (\Delta^i)_{kj} \partial_k V$

A possible solution is for quadratic potentials : $V = \sum_{ij} \Delta_{ij} x_i x_j$

|| But otherwise there is no generic
characterisation of equilibrium with unrelated noise.

Generalisation to several coupled noises: continuum

Lecture on
Complex Systems (1.18)



$$\langle \eta_i(t) \eta_j(t') \rangle = 2\delta_{ij} \delta(t-t') \longrightarrow \langle \eta(y, t) \eta(y', t') \rangle = \underbrace{R(y-y')}_{\text{describes correlations in direction } y} \delta(t-t')$$

$(R(y) = 2D \delta(y))$ for uncorrelated noise]

$P(x, t) \longrightarrow P[\rho, t]$ (functional) probability density of the profile $\rho(y)$, at time t

* Langevin equation: from $\partial_t x_i = F_i(x, t) + \eta_i$ (see p. 1.16) one goes to

$$\boxed{\partial_t \rho(y, t) = \mathcal{F}[\rho; y, t] + \eta(y, t)}$$

* Fokker-Planck equation: from $\partial_t P(x, t) = - \sum_i \partial_i (F_i(x, t) P(x, t)) + \sum_{i,j} \Delta_{ij} \partial_i \partial_j P(x, t)$

$$\boxed{\partial_t P[\rho, t] = - \int dy \frac{\delta}{\delta \rho(y)} (\mathcal{F}[\rho; y, t] P[\rho, y]) + \frac{1}{2} \int dy dy' R(y-y') \frac{\delta^2 P[\rho, t]}{\delta \rho(y) \delta \rho(y')}}$$

* Ito formula:

$$\partial_t (\varphi(\rho, t)) = \partial_t \varphi(\rho, t) + \int dy \mathcal{F}(\rho; y, t) \frac{\delta \varphi(\rho, t)}{\delta \rho(y)} + \frac{1}{2} \int dy dy' R(y-y') \frac{\delta^2 \varphi(\rho, t)}{\delta \rho(y) \delta \rho(y')}$$

* Example of equilibrium (i.e. reversible) dynamics: $\mathcal{F}(\rho; y, t) = \frac{\delta H(\rho)}{\delta \rho(y)}$

Then for an uncorrelated noise $R(y) = 2T \delta(y)$

$$\boxed{P_{eq}[\rho] \propto \exp\left(-\frac{1}{T} H(\rho)\right)}$$
 is a solution of the FP equation, with current:

$$-\mathcal{F}(\rho; y, t) P_{eq}[\rho] + \frac{1}{2T} \frac{\delta P_{eq}[\rho]}{\delta \rho(y)} = 0$$

• Generalization to position-dependent noise:

(1.19)
lecture on
Complex Systems

But writing:

$$D(x,t) > 0$$

$$\boxed{\partial_t x = F(x,t) + \sqrt{2D(x,t)} \eta(t)} \quad \text{with } \langle \eta(t) \eta(t') \rangle = \delta(t-t')$$

Fokker-Planck: $\text{Prob}(x(t)) = \exp\left(-\frac{1}{2} \int_0^t \frac{(\partial_x x - F(x,t))^2}{2D(x,t)} dt\right)$

$$\boxed{\partial_t P(x,t) = -\partial_x (F(x,t) P(x,t)) + \partial_x^2 (D(x,t) P(x,t))}$$

• With several coupled noises:

$$\boxed{\partial_t x_i = F_i(x,t) + \sqrt{2D(x,t)} \eta_i(t)} \quad \langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t-t')$$

$$\boxed{\partial_t P(x,t) = - \sum_i \partial_i (F_i(x,t) P(x,t)) + \sum_{ij} \partial_i \partial_j (\Delta_{ij} D(x,t) P(x,t))}$$

• Generalization to fields: $x_i(t) \mapsto \rho(y,t)$

$$\partial_t \rho(y,t) = F(\rho(y,t),t) + \sqrt{2D(\rho(y,t))} \eta(y,t) \quad \langle \eta(y,t) \eta(y',t') \rangle = R(y-y') \delta(t-t')$$

$$\partial_t P[\rho(y),t] = - \int dy \frac{\delta}{\delta \rho(y)} \left(F(\rho(y),t) P[\rho(y),t] \right) + \int dy dy' \frac{\delta^2}{\delta \rho(y) \delta \rho(y')} \left[R(y-y') P[\rho(y),t] \right]$$

Remark : Backward Fokker-Planck equation:

(1.20)

Lecture on
Complex Systems

We have seen that $P(x, t | x_0, t_0) = P(x, t)$ verifies the forward FP equation

$$\partial_t P(x, t) = -\partial_x (F(x, t) P(x, t)) + D \partial_x^2 P(x, t)$$

We might be interested in the derivatives $\partial_{x_0}, \partial_{t_0}$ w.r.t. the initial conditions (e.g. to tackle 1st passage problems).

As on page 110, we integrate the Langevin equation $\dot{x} = +F(x, t) + \eta$ between $t_0 - \delta t$ and t_0 to write $x(t_0) = \underbrace{x(t_0 - \delta t)}_{= x_0} + \delta t F(x_{-1}, t - \delta t) + \int_{t_0 - \delta t}^{t_0} d\tau \eta(\tau)$ with η Gaussian, $\langle \eta \rangle = 0$, $\langle \eta^2 \rangle = 2D\delta t$

Then, for any function $f(x)$ one has:

$$\int dx f(x) P(x, t | x_0, t_0) = \int dx f(x) \int dy P(y) \underbrace{P(x, t | x_0 - \eta - \delta t F(x_0, t), t_0 - \delta t)}_{\text{or } F(x_0, t - \delta t) \text{ but this is the same at the order } \delta t \text{ we are interested in}}$$

we perform the average $\int dy P(y)$ through $\langle \eta \rangle = 0$, $\langle \eta^2 \rangle = 2D\delta t$

$$\begin{aligned} &\approx P(x, t | x_0, t_0 - \delta t) - \eta \partial_{x_0} P(x, t | x_0, t_0) \\ &- \delta t \partial_{x_0} P(x, t | x_0, t_0) - \frac{1}{2} \eta^2 \partial_{x_0} P(x, t | x_0, t_0) + O(\delta t^{3/2}) \\ &F(x_0, t) \quad \text{one has to expand up to order } \delta t \end{aligned}$$

Collecting:

$$\int dx f(x, t) \underbrace{\frac{P(x, t | x_0, t_0) - P(x, t | x_0, t_0 - \delta t)}{\delta t}}_{\rightarrow \partial_{t_0} P(x, t | x_0, t_0)} = \int dx f(x, t) \left[-F(x_0, t) \partial_{x_0} P(x, t | x_0, t_0) - D \partial_{x_0}^2 P(x, t | x_0, t_0) \right]$$

And finally:

$$-\partial_{t_0} P(x, t | x_0, t_0) = F(x_0, t) \partial_{x_0} P(x, t | x_0, t_0) + D \partial_{x_0}^2 P(x, t | x_0, t_0)$$

Note the change of sign & the position of ∂_{x_0}

This is the backwards Fokker-Planck equation, which describes the variations of the probability density with respect to its initial condition.

$$P(x, t | x_0, t_0) = e^{\frac{(t-t_0)}{D} W} P_0(x_0)$$

↑
indep. of time

For homogeneous processes where $P(x, t | x_0, t_0)$ is a function of $t - t_0$ only, e.g. if $F(x, t) = F(x)$ one has $\partial_{t_0} = -\partial_t$ and hence

$$\partial_t P(x, t | x_0, t_0) = F(x_0) \partial_{x_0} P(x, t | x_0, t_0) + D \partial_{x_0}^2 P(x, t | x_0, t_0)$$