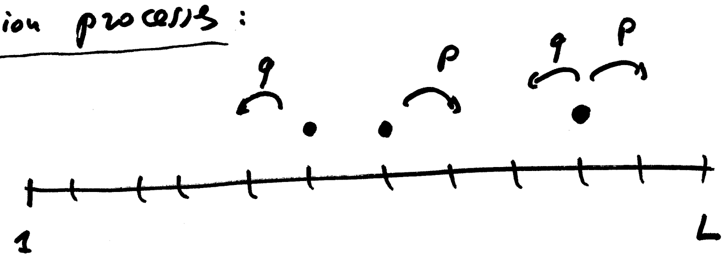
Lecture 2 - 30/11/2011

- **Motivations :**

Exclusion process :



× Each site i is either empty ($n_i = 0$) or occupied ($\bullet$, $n_i = 1$) by a particle
Each particle can jump to its left (right) with rate q (p)
provided the target site is empty (EXCLUSION RULE)

× This can model :
- car / pedestrian (in a queue) traffic, jam
- motion of motor proteins (e.g.: dynein, kinesin) on microtubules

× Classification :

SSEP	(Symmetric)	$p = q$
ASEP	(asymmetric)	$p \neq q$
TASEP	(totally -)	$q = 0$ $p > 0$
WASEP	(weakly asymmetric)	$p - q = \frac{E}{L}$

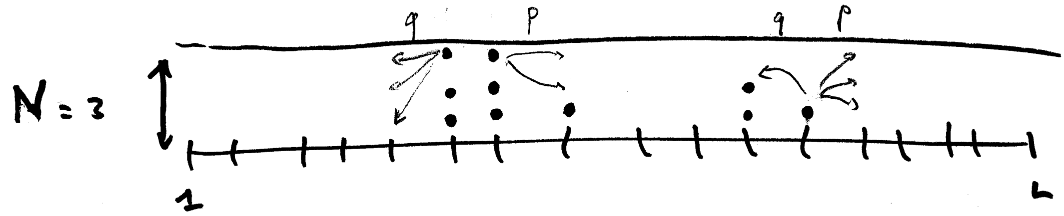
× This is the "Ising model" of non-equilibrium dynamics
The (non-equilibrium) steady state (NESS) is known in many situations
- closed boundary condition, or periodic
- contact with reservoirs at the boundaries

The large deviation of current / activity have been computed in the large size limit (with finite size correction) in the (WA)SEP.

- Partial Exclusion Process :

The exclusion condition is relaxed : each site can contain between 0 and $N<\infty$ particles

Each particle can jump to any of the empty site with rate p or q



in other words, in terms of particle number transition rates:

p ↷ (jump to the right) choice of the moving particle

$$\boxed{\begin{aligned} W(\dots n_i\, n_{i+1} \dots \to \dots n_i-1\;\, n_{i+1}+1 \dots) &= p\; n_i\, (N-n_{i+1}) \\ W(\dots n_i\, n_{i+1} \dots \xrightarrow{q} \dots n_i+1\;\, n_{i+1}-1 \dots) &= q\,(N-n_i)\; n_{i+1} \end{aligned}}$$

← choice of the target site

← particle moving

target site

This implements the partial exclusion ($N-n=0$ if the occupation # n of the target site is $n=N$ i.e. if the target site is full)

- Spin operators for stochastic processes :

One defines, for each site described by a vector $|n\rangle$ $(0\le n\le N)$

$$\boxed{\begin{aligned} S^+|n\rangle &= (N-n)\,|n+1\rangle \\ S^-|n\rangle &= n\,|n-1\rangle \\ \hat{n}\,|n\rangle &= n\,|n\rangle \end{aligned}}$$

or matricially: $\hat{n} = \begin{pmatrix} 0 & & (0) \\ & \ddots & \\ (0) & & N \end{pmatrix}$

$S^+ = \begin{pmatrix} 0 & & & (0) \\ N & \ddots & & \\ & \ddots & \ddots & \\ (0) & & 1 & 0 \end{pmatrix}$ $\quad S^- = \begin{pmatrix} 0 & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & N \\ (0) & & & 0 \end{pmatrix}$

One also reads from the matrices :

$$\left.\begin{aligned} \langle n|S^+ &= \langle n-1|\,(N-n+1) \\ \langle n|S^- &= \langle n+1|\,(n+1) \end{aligned}\right\}$$

where by convention one notes :

$$\langle -1| = \langle N+1| = 0$$

- Commutation relations:

One defines $S^z = \hat{n} - \frac{1}{2}N = \begin{pmatrix} -\frac{N}{2} & & (0) \\ & \ddots & \\ (0) & & \frac{N}{2} \end{pmatrix}$

One checks directly that $S^\pm, S^z$ obey the commutation relations

$$\boxed{[S^z, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = 2S^z}$$

which are the same for quantum operators

Similarly, defining $S^x = \frac{1}{2}(S^+ + S^-)$, $S^y = \frac{1}{2i}(S^+ - S^-)$ one has $\boxed{[S^x, S^y] = i S^z}$ and similar cyclic permutation (i: complex number $i^2 = -1$)

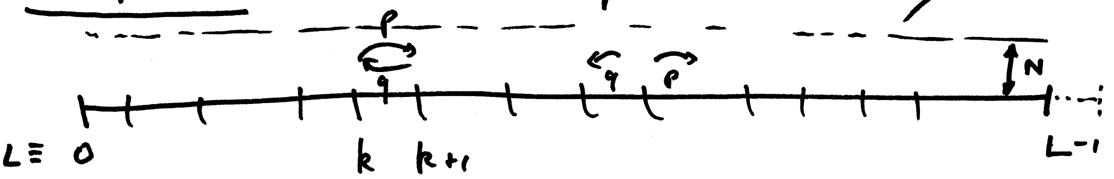
Let's e.g show that $[S^+, S^-] = 2S^z = 2\hat{n} - N$:

$$[S^+, S^-]|n\rangle = S^+ n|n-1\rangle - S^-(N-n)|n+1\rangle = \big[(N-n+1)n - (N-n)(n+1)\big]|n\rangle$$

$$Nn - n^2 + n - Nn - N + n^2 + n$$

$$= (2n - N)|n\rangle \text{ hence the result}$$

- Example 1: ASEP with periodic boundary conditions



$$\partial_t P(\vec{n}, t) = \sum_{k=0}^{L-1} \Big\{ p(n_k+1)(N-n_{k+1}-1) P(n_k+1, n_{k+1}-1, t) + q(N-n_k+1)(n_{k+1}+1) P(n_k-1, n_{k+1}+1, t)$$
$$- q\, n_{k+1}(N-n_k) P(\vec{n}) - p\, n_k(N-n_{k+1}) P(\vec{n}) \Big\}$$

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \sum_{k=0}^{L-1} \Big\{ p \underbrace{(n_k+1)(N-n_{k+1}-1) P(n_k+1, n_{k+1}-1, t)}_{n_k \mapsto n_k - 1 \,;\, n_{k+1} \mapsto n_{k+1}+1} |\vec{n}\rangle + q \underbrace{(N-n_k+1)(n_{k+1}+1) P(n_k-1, n_{k+1}-1, t)}_{n_k \mapsto n_k+1 \quad n_{k+1} \mapsto n_{k+1}-1} |\vec{n}\rangle$$
$$- q\, n_{k+1}(N-n_k) P(\vec{n})|\vec{n}\rangle - p\, n_k(N-n_{k+1}) P(\vec{n})|\vec{n}\rangle \Big\}$$

change of variable on $\vec{n}$ so as to factorize by $P(\vec{n})$

$$= \sum_{\vec{n}} \sum_{k=0}^{L-1} \Big\{ p\, n_k(N-n_{k+1}) |n_k-1, n_{k+1}+1\rangle + q(N-n_k) n_{k+1} |n_k+1, n_{k+1}-1\rangle$$
$$- p\, n_k(N-n_{k+1}) |\vec{n}\rangle - q(N-n_k) n_{k+1} |\vec{n}\rangle \Big\} P(\vec{n})$$

in the sums: $L \equiv 0$ (periodic b.c.)

In the gain term of that equation, one reckognizes

$$\begin{cases} n_k (N-n_{k+1}) \, | n_k-1, n_{k+1}+1 \rangle = S_k^- S_{k+1}^+ \, |\vec{n}\rangle \\ (N-n_k) n_{k+1} \, | n_k+1, n_{k+1}-1 \rangle = S_k^+ S_{k+1}^- \, |\vec{n}\rangle \end{cases}$$

one notes $\hat{n}_k : \hat{n}_k|\vec{n}\rangle = n_k$

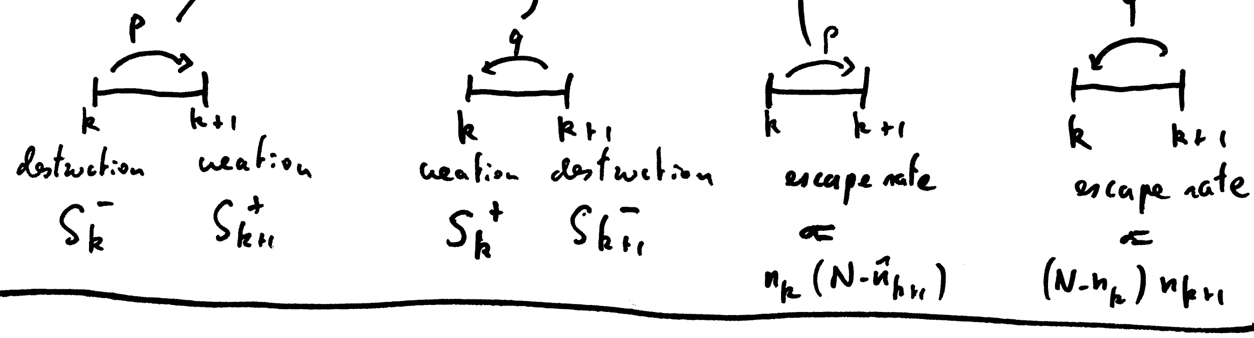
$\boxed{\check{n}_k = N - \hat{n}_k}$

<u>Thus:</u>

$$\partial_t |P(t)\rangle = \sum_{k=0}^{L-1} \left\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\} \sum_{\vec{n}} P(\vec{n}) |\vec{n}\rangle$$

One identifies $\mathbb{W}$ in $\partial_t |P(t)\rangle = \mathbb{W} |P(t)\rangle$ to:

$$\boxed{\mathbb{W} = \sum_{k=0}^{L-1} \left\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \hat{n}_k \check{n}_{k+1} \right\}}$$



One thus reads directly from the dynamics the form of $\mathbb{W}$

- <u>Example 1 bis</u>: s-modified operator of evolution $\mathbb{W}(s)$

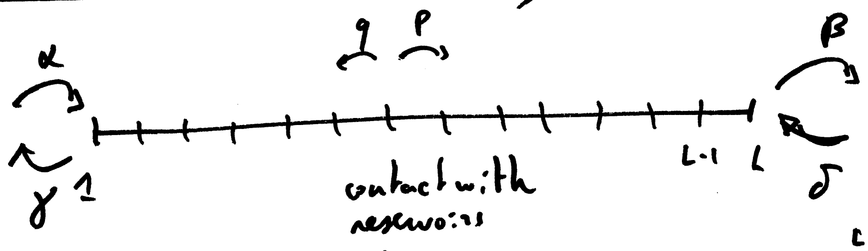
  × <u>s conjugated to the activity K</u>:

$$\mathbb{W}(s) = \sum_{k=0}^{L-1} \left\{ e^{-s} \left[p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- \right] - p\, \hat{n}_k \check{n}_{k+1} - q\, \hat{n}_k \check{n}_{k+1} \right\}$$

  × <u>s conjugated to the total current Q</u>:

$$\mathbb{W}(s) = \sum_{k=0}^{L-1} \left\{ e^{-s} p\, S_k^- S_{k+1}^+ + e^{s} q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \hat{n}_k \check{n}_{k+1} \right\}$$

- Example 2 : with boundary conditions with reservoirs



$$\mathbb{W} = \mathbb{W}_{bulk} + \overbrace{\mathbb{W}_{res}} \quad \text{where} \quad \mathbb{W}_{bulk} = \sum_{k=1}^{L-1} \left\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\}$$

× from the rules given previously one expects:

$$\mathbb{W}_{res} = \alpha(S_1^+ - \check{n}_1) + \delta(S_L^+ - \check{n}_L) + \gamma(S_1^- - \hat{n}_1) + \beta(S_L^- - \hat{n}_L)$$

× let's for instance check explicitly the left $\binom{\alpha}{\gamma}$ boundary term:

$$\partial_t \mathbb{P}(\vec{n},t) \overset{\alpha\gamma}{=} \alpha(N-n_1+1)P(n_1-1,t) + \gamma(n_1+1)P(n_1+1,t) - \alpha(N-n_1)P(n_1,t) - \gamma\, n_1\, P(n_1,t)$$

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \Big(\overbrace{\alpha(N-n_1+1)P(n_1-1,t)|\vec{n}\rangle}^{n_1 \mapsto n_1+1} + \overbrace{\gamma(n_1+1)P(n_1+1,t)|\vec{n}\rangle}^{n_1 \mapsto n_1-1} - \alpha(N-n_1)P(n_1,t)|\vec{n}\rangle - \gamma\, n_1\, P(n_1,t)|\vec{n}\rangle \Big)$$

change of variable on $\vec{n}$ so as to factorise $P(\vec{n},t)$

$$= \sum_{\vec{n}} \Big(\overbrace{\alpha(N-n_1)|n_1+1\rangle}^{S_1^+|\vec{n}\rangle} + \overbrace{\gamma\, n_1 |n_1-1\rangle}^{S_1^-|\vec{n}\rangle} - \alpha(N-n_1)|\vec{n}\rangle - \gamma\, n_1 |\vec{n}\rangle \Big) P(\vec{n},t)$$

$$= \underbrace{\Big(\alpha(S_1^+ - \check{n}_1) + \gamma(S_1^- - \hat{n}_1) \Big)}_{\text{this is the left term of the boundary operator } \mathbb{W}_{res} \text{ that we expected}} \underbrace{\sum_{\vec{n}} P(\vec{n},t)|\vec{n}\rangle}_{|P(t)\rangle}$$

• Analogy with quantum mechanics

What is the link between the $S^\pm$, S^z and the quantum ones?

. Quantum operators: $\Sigma^\pm$, Σ^z verify

$$\begin{cases} \Sigma^+|n\rangle = \sqrt{(n+1)(N-n)}\;|n+1\rangle \\ \Sigma^-|n\rangle = \sqrt{n(N-n+1)}\;|n-1\rangle \\ \Sigma^z|n\rangle = (n-\tfrac{1}{2}N)\;|n\rangle \end{cases} \quad \to \quad \Sigma^z = S^z$$

Besides, they are hermitian-adjoint: $(\Sigma^+)^\dagger = \Sigma^-$

× From quantum to statistical operators:

Consider $Q = \binom{N}{\hat{n}}^{-1/2}$ (binomial coefficient) $\binom{N}{n} = \dfrac{N!}{n!\,(N-n)!}$

and let's show that

$$\boxed{S^\pm = Q^{-1}\Sigma^\pm Q}$$

$$\frac{(n+1)!\,(N-n-1)!}{n!\,(N-n)!} = \frac{n+1}{N-n}$$

. $$Q^{-1}\Sigma^+ Q|n\rangle = \binom{N}{n}^{-1/2} Q^{-1}\sqrt{(n+1)(N-n)}\;|n+1\rangle = \sqrt{(n+1)(N-n)}\left(\frac{\binom{N}{n}}{\binom{N}{n+1}}\right)^{-1/2}|n+1\rangle$$

$$= \sqrt{\frac{(n+1)(n+1)^{-1}(N-n)}{(N-n)^{-1}}}\;|n+1\rangle = (N-n)\,|n+1\rangle = S^+|n\rangle \quad \text{ok}$$

. $$Q^{-1}\Sigma^- Q|n\rangle = \binom{N}{n}^{-1/2} Q^{-1}\sqrt{n(N-n+1)}\;|n-1\rangle = \sqrt{n(N-n+1)}\left(\frac{\binom{N}{n}}{\binom{N}{n-1}}\right)^{-1/2}|n-1\rangle$$

$$= \sqrt{\frac{n(N-n+1)n}{N-n+1}}\;|n-1\rangle \qquad \left(\frac{n!\,(N-n)!}{(n-1)!\,(N-n+1)!}\right)^{-1} = \left(\frac{n}{N-n+1}\right)^{-1}$$

$$= n\,|n-1\rangle = S^-|n-1\rangle \quad \text{OK}$$

× To summarize: $S^\pm$ are (non-unitary) similarity transformations through Q of the quantum spin operators $\Sigma^\pm$

Since the transformation is not unitary: $(S^+)^\dagger \neq S^-$ while $(\Sigma^+)^\dagger = \Sigma^-$

Indeed $(S^+)^\dagger = Q^\dagger \Sigma^- Q^{-1\dagger} = \underbrace{Q^\dagger Q}_{\neq \mathbb{1}}\, S^-\, \underbrace{Q^{-1} Q^{-1\dagger}}_{\neq \mathbb{1}}$ since Q is not unitary lecture on stoch. processes (2.7)

• <u>Steady state</u> in equilibrium for the symmetric S.E.P.

Fix a density $0<\rho<1$ and consider the eq. distrib. product of Bernoulli laws

$$P_{eq}^{\rho}(\vec{n}) = \prod_{k=1}^{L} \binom{N}{n_k} \rho^{n_k} (1-\rho)^{N-n_k}$$

× case of a single site : $|B^\rho\rangle = \sum_{n=0}^{N} P_{eq}^{\rho}(n)|n\rangle = \sum_{n=0}^{N} \binom{N}{n} \rho^{n}(1-\rho)^{N-n}|n\rangle$

$$(N-n+1)\frac{N!}{(n-1)!(N-n+1)!} = n\frac{N!}{n!(N-n)!}$$

action of S^+ :

$$S^+|B^\rho\rangle = \sum_{n=0}^{N}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}(N-n)|n+1\rangle \overset{n\to n-1}{=} \sum_{n=0}^{N}\binom{N}{n-1}\rho^{n-1}(1-\rho)^{N-n+1}(N-n+1)|n\rangle$$

$$= \sum_{n=0}^{N} n\,\frac{1-\rho}{\rho}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}|n\rangle \quad \text{we used } (N-n+1)\binom{N}{n-1} = n\binom{N}{n}$$

$$= \frac{1-\rho}{\rho}\sum_{n=0}^{N}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}\underbrace{n|n\rangle}_{\hat{n}|n\rangle} = \frac{1-\rho}{\rho}\hat{n}|B^\rho\rangle$$

$$\boxed{S^+|B^\rho\rangle = \frac{1-\rho}{\rho}\hat{n}|B^\rho\rangle}$$

$$(n+1)\frac{N!}{(n+1)!(N-n-1)!} = (N-n)\frac{N!}{n!(N-n)!}$$

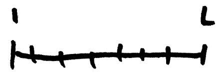
action of S^- :

$$S^-|B^\rho\rangle = \sum_{n=0}^{N}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}n|n-1\rangle = \sum_{n=0}^{N}\binom{N}{n+1}\rho^{n+1}(1-\rho)^{N-n-1}(n+1)|n\rangle$$

$$= \sum_{n=0}^{N}(N-n)\left(\frac{1-\rho}{\rho}\right)^{-1}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}|n\rangle \quad \text{where we used } (n+1)\binom{N}{n+1} = (N-n)\binom{N}{n}$$

$$= \frac{\rho}{1-\rho}\sum_{n=0}^{N}\binom{N}{n}\rho^{n}(1-\rho)^{N-n}(N-n)|n\rangle = \frac{\rho}{1-\rho}(N-\hat{n})|B^\rho\rangle$$

$$\boxed{S^-|B^\rho\rangle = \frac{\rho}{1-\rho}\check{n}|B^\rho\rangle}$$

× case of  with p.b.c and $p=q$: $\mathbb{W} = \sum_{k=1}^{L}\{S_k^+S_{k+1}^- + S_k^-S_{k+1}^+ - \hat{n}_k\check{n}_{k+1} - \check{n}_k\hat{n}_{k+1}\}$

Product law on the sites

Thus $\mathbb{W}|B^\rho\rangle = \sum_{k=1}^{L}\left\{\underbrace{\frac{\rho}{1-\rho}\frac{1-\rho}{\rho}}_{=1}\hat{n}_k\check{n}_{k+1} + \underbrace{\frac{\rho}{1-\rho}\frac{1-\rho}{\rho}}_{=1}\hat{n}_{k+1}\check{n}_k - \hat{n}_k\check{n}_{k+1} - \check{n}_k\hat{n}_{k+1}\right\}|B^\rho\rangle$

and one finds $\mathbb{W}|B^\rho\rangle = 0$: $\boxed{|B^\rho\rangle \text{ is the steady state with } \rho \text{ fixed by the total \# of particles}}$

One should constrain total # of particles →

- Eq. steady state in contact with reservoirs:

Now, the # of particle is not fixed.
Let's see how $\alpha\, \beta\, \gamma\, \delta$ are determined, and fix ρ:

$$\mathbb{W}_{\text{reservoirs}} = \alpha\left(S_1^+ - \check{n}_1\right) + \delta\left(S_L^+ - \check{n}_L\right) + \gamma\left(S_1^- - \hat{n}_1\right) + \beta\left(S_L^- - \hat{n}_L\right)$$

$$\mathbb{W}_{\text{res}} |B\rho\rangle = \alpha\left(\frac{1-\rho}{\rho}\hat{n}_1 - \check{n}_1\right) + \delta\left(\frac{1-\rho}{\rho}\hat{n}_L - \check{n}_L\right) + \gamma\left(\frac{\rho}{1-\rho}\check{n}_1 - \hat{n}_1\right) + \beta\left(\frac{\rho}{1-\rho}\check{n}_L - \hat{n}_L\right)$$

$$= \left(\alpha\frac{1-\rho}{\rho} - \gamma\right)\hat{n}_1 + \left(\delta\frac{1-\rho}{\rho} - \beta\right)\hat{n}_L + \left(\gamma\frac{\rho}{1-\rho} - \alpha\right)\underbrace{\check{n}_1}_{(N-\hat{n}_1)} + \left(\beta\frac{\rho}{1-\rho} - \delta\right)\underbrace{\check{n}_L}_{(N-\hat{n}_L)}$$

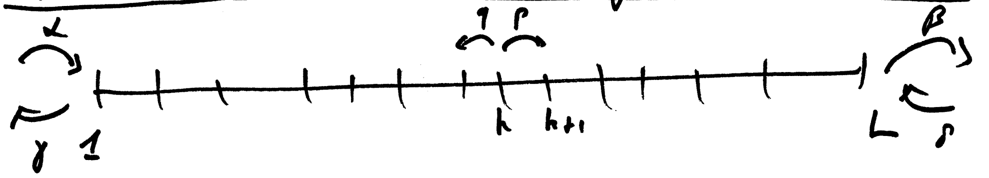
For all those terms to vanish one needs:

$$\begin{cases} \alpha\dfrac{1-\rho}{\rho} = \gamma \\[2mm] \delta\dfrac{1-\rho}{\rho} = \beta \end{cases} \qquad \text{ie} \qquad \boxed{\begin{array}{c} \rho = \dfrac{\alpha}{\alpha+\gamma} \\ \hline \rho = \dfrac{\delta}{\delta+\beta} \end{array}}$$

equality implies $\dfrac{\alpha}{\gamma} = \dfrac{\delta}{\beta}$

$$\boxed{\text{The solution for the steady state is thus } |B\rho\rangle \text{ with } \rho = \frac{\alpha}{\alpha+\gamma} = \frac{\delta}{\delta+\beta} \text{ provided } \frac{\alpha}{\gamma} = \frac{\delta}{\beta}}$$

- Symmetry 1 : conservation of particle number :



Let's relate the operator $\mathbb{W}_{tot}(s)$ for the total current to the one $\mathbb{W}_1(s)$ for the current at the right boundary :

$$\mathbb{W}_{tot}(s) = \sum_{k=1}^{L-1} \Big\{ p\, S_k^- S_{k+1}^+ \underbrace{e^{-s}}_{\text{jump} \rightarrow} + q\, S_k^+ S_{k+1}^- \underbrace{e^{s}}_{\text{jump} \leftarrow} - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \Big\} + \alpha\left(e^{-s} S_1^+ - \check{n}_1\right) + \delta\left(S_L^+ e^{s} - \check{n}_L\right) + \gamma\left(e^{s} S_1^- - \hat{n}_1\right) + \beta\left(S_L^- e^{-s} - \hat{n}_L\right)$$

× Let's find a similarity transformation Q such that $Q^{-1} S^{\pm} Q = (z)^{\pm 1} S^{\pm}$:

$$\left.\begin{aligned} z^{+\hat{n}} S^+ z^{-\hat{n}} |n\rangle &= z^{-n} z^{(n+1)} (N-n)|n+1\rangle = z\, S^+|n\rangle \\ z^{-\hat{n}} S^- z^{\hat{n}} |n\rangle &= z^{n} z^{-(n+1)}\, n\, |n-1\rangle = z^{-1} S^-|n\rangle \end{aligned}\right\} \text{ thus } \underline{Q = z^{-\hat{n}} \text{ works}}$$

× Apply $Q^{-1} \cdot Q$ with $Q = e^{-s \sum_{k=1}^{L} k \hat{n}_k}$:

$$\text{using } \left| \begin{aligned} Q^{-1} S_k^- S_{k+1}^+ Q &= e^{-sk} e^{s(k+1)} S_k^- S_{k+1}^+ = e^{s} S_k^- S_{k+1}^+ \\ Q^{-1} S_k^+ S_{k+1}^- Q &= e^{sk} e^{-s(k+1)} S_k^+ S_{k+1}^- = e^{-s} S_k^+ S_{k+1}^- \end{aligned} \right| \begin{aligned} Q^{-1} S_1^{\pm} Q &= e^{\pm s} S_1^{\pm} \\ Q^{-1} S_L^{\pm} Q &= e^{\pm Ls} S_L^{\pm} \end{aligned}$$

One finds :

$$Q^{-1} \mathbb{W}_{tot}(s) Q = \overbrace{\sum_{k=1}^{L-1} \Big\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \Big\}}^{s \text{ has disappeared from the bulk}} + \alpha\left(S_1^+ - \check{n}_1\right) + \overbrace{\delta\left(e^{s(L+1)} S_L^+ - \check{n}_L\right)}^{\text{and remains at the right boundary}} + \gamma\left(S_1^- - \hat{n}_1\right) + \beta\left(e^{-s(L+1)} S_L^- - \hat{n}_L\right)$$

$$\boxed{Q^{-1} \mathbb{W}_{tot}(s) Q = \mathbb{W}_1\big((L+1)s\big)}$$

and the same relation thus holds for the large deviation function

$$\boxed{\psi_{tot}(s) = \psi_1\big((L+1)s\big)}$$

This means that determining the statistics of the total current is equivalent to determining the statistics of the current to the right reservoir, which is in general more simple.

• Symmetry 2 : invariance by rotation

12.10 lecture on stoch. processes

in periodic boundary condition : let's explicit the S^{xyz} operators :

$$\left| \begin{array}{l} S^{\pm} = S^x \pm i S^y \quad \text{ie: } S^x = \dfrac{S^+ + S^-}{2} \qquad S^y = \dfrac{S^+ - S^-}{2i} \\ \hat{n} = \dfrac{N}{2} + S^z \\ \check{n} = \dfrac{N}{2} - S^z \end{array} \right.$$

$p = q = 1$

$$\mathbb{W} = \sum_{k=1}^{L} \left\{ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \check{n}_k \hat{n}_{k+1} \right\}$$

– idem $k \to k+1$

$$= \sum_{k=1}^{L} \left\{ (S_k^x + i S_k^y)(S_{k+1}^x - i S_{k+1}^y) + (S_k^x - i S_k^y)(S_{k+1}^x + i S_{k+1}^y) - \left(\frac{N}{2} + S_k^z\right)\left(\frac{N}{2} - S_k^z\right) \right.$$

$$2\Big\{ \underbrace{S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + S_k^z S_{k+1}^z}_{\text{denoted by } \vec{S}_k \cdot \vec{S}_{k+1}} + \text{cste} \Big\}$$

$$\boxed{\mathbb{W} = 2 \sum_{k=1}^{L} \vec{S}_k \cdot \vec{S}_{k+1}} \qquad \vec{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix}$$

R : matrix of rotation in 3D . $R\vec{S}$ is again a set of 3 spin operators

__Besides__ : $(R\vec{S}_k) \cdot (R\vec{S}_{k+1}) = \vec{S}_k \cdot \vec{S}_{k+1}$ (invariance by rotation of the scalar product)

One thus have an invariance by rotation of the spin operators $\vec{S}$

It does not affect the bulk part of $\mathbb{W}$ to apply R.

• __Use__ : for $\mathbb{W}_1(s)$ in open boundary conditions, R modifies those boundary terms.
After the rotation one can reinterpret them, for $s \neq 0$ as systems in contact with different densities (eg. at eq.)
→ $\boxed{\text{Mapping btw equilibrium and non-equilibrium}}$