- Large deviation functions as maximal eigenvalues

Lecture on Stoch. processes (2.10)

1 - Cases of observables of type A_1

× Time evolution for $P(\mathcal{C}, A_1, t)$:

$$\partial_t P(\mathcal{C}, A_1, t) = \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C})\, P(\mathcal{C}', A_1 - a_1(\mathcal{C}' \to \mathcal{C}), t) - r(\mathcal{C})\, P(\mathcal{C}, A_1, t)$$

this evolution is non-diagonal in the direction A_1
the operator of evolution is very difficult to diagonalize.

× Going to the s-state: Laplace transform

One introduces $\underbrace{\hat{P}(\mathcal{C}, s, t) = \sum_{A_1} e^{-sA_1} P(\mathcal{C}, A_1, t)}$ This quantity is more detailed than $\langle e^{-sA_1} \rangle$, but it obeys a closed equation of evolution, as detailed now.

It verifies $\langle e^{-sA_1} \rangle = \sum_{\mathcal{C}, A_1} e^{-sA_1} P(\mathcal{C}, A_1, t) = \sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t)$

× Time evolution: from the time evolution of $P(\mathcal{C}, A_1, t)$ one finds:

$$\partial_t \hat{P}(\mathcal{C}, s, t) = \sum_{\mathcal{C}'} e^{-s a_1(\mathcal{C}' \to \mathcal{C})} W(\mathcal{C}' \to \mathcal{C})\, \hat{P}(\mathcal{C}', s, t) - r(\mathcal{C})\, \hat{P}(\mathcal{C}, s, t)$$

This time, the operator is diagonal in direction s. Putting $|\hat{P}(t)\rangle = \sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t)|\mathcal{C}\rangle$

one has

$$\boxed{\begin{aligned} \partial_t |\hat{P}(s,t)\rangle &= \mathbb{W}(s)\, |\hat{P}(s,t)\rangle \\ (\mathbb{W}(s))_{\mathcal{C}\mathcal{C}'} &= \underbrace{e^{-s a_1(\mathcal{C}' \to \mathcal{C})}}\, W(\mathcal{C}' \to \mathcal{C}) - r(\mathcal{C})\, \delta_{\mathcal{C}\mathcal{C}'} \end{aligned}}$$

here, only the non-diagonal part of the evolution operator is modified by s: it is the part describing jumps

× Largest eigenvalue of $\mathbb{W}(s)$: one has $\langle e^{-sA_1} \rangle = \sum_{\mathcal{C}} \underbrace{\hat{P}(\mathcal{C}, s, t)}$

but $|\hat{P}(s,t)\rangle = e^{t\mathbb{W}(s)} |\hat{P}_0\rangle \underset{t\to\infty}{\sim} e^{t \underbrace{\max \mathrm{Sp}\, \mathbb{W}(s)}_{\text{maximal eigenvalue of } \mathbb{W}(s)}} \Rightarrow \sim e^{t \max \mathrm{Sp}\, \mathbb{W}(s)}$

Hence $\langle e^{-sA_1} \rangle \sim e^{t \max \mathrm{Sp}\, \mathbb{W}(s)}$ and by definit° $\boxed{\psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)}$ it is the largest eigenvalue of $\mathbb{W}(s)$

2. Case of observable of type A_2

$\partial_t P(\mathcal{C}, A_2, t)$ is not so directly obtained as previously.

* first approach: time discretization

OK up to order dt

$$P(\mathcal{C}, A_2, t+dt) = \sum_{\mathcal{C}'} dt\, W(\mathcal{C}' \to \mathcal{C})\, P(\mathcal{C}', A_2, t) + (1 - dt\, r(\mathcal{C}))\, \underbrace{P\Big(\mathcal{C},\ A_2 - \overbrace{\int_t^{dt+t} d\tau\, a_2(\mathcal{C}(\tau))}^{\simeq dt\, a_2(\mathcal{C})}\Big)}_{P(\mathcal{C}, A_2, t) - dt\, \partial_{A_2} P(\mathcal{C}, A_2, t)\, a_2(\mathcal{C})}$$

Hence, in the $dt \to 0$ limit:

$$\boxed{\partial_t P(\mathcal{C}, A_2, t) = \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C}) P(\mathcal{C}', A_2, t) - \Big(r(\mathcal{C}) P(\mathcal{C}, A_2, t) + a_2(\mathcal{C}) \partial_{A_2} P(\mathcal{C}, A_2, t)\Big)}$$

Or, introducing $\hat{P}(\mathcal{C}, s, t) = \int dA_2\, e^{-sA_2} P(\mathcal{C}, A_2, t)$

one again has $\langle e^{-sA_2} \rangle = \int dA_2 \sum_{\mathcal{C}} P(\mathcal{C}, A_2, t) = \sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t)$

Thanks to Laplace transform, the derivative ∂_{A_2} becomes a simple factor

The equation of evolution is thus (by integration by part)

$$\partial_t \hat{P}(\mathcal{C}, s, t) = \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C}) \hat{P}(\mathcal{C}', s, t) - \big(r(\mathcal{C}) + s\, a_2(\mathcal{C})\big) \hat{P}(\mathcal{C}, s, t)$$

Or, in vector notation:

$$\boxed{\begin{aligned} \partial_t |\hat{P}(s,t)\rangle &= \mathbb{W}(s)\, |\hat{P}(s,t)\rangle \\ \big(\mathbb{W}(s)\big)_{\mathcal{C}\mathcal{C}'} &= W(\mathcal{C}' \to \mathcal{C}) - \underbrace{\big(r(\mathcal{C}) + s\, a_2(\mathcal{C})\big) \delta_{\mathcal{C}\mathcal{C}'}} \end{aligned}}$$

this time, it is the diagonal part of the operator of evolution which is modified by s

This corresponds to the fact that $A_2 = \int_0^t d\tau\, a_2(\mathcal{C}(\tau))$ is an observable which does not evolve <u>at</u> jumps.

<u>Remark</u>: This formula, written as:

$$\partial_t |\hat{P}(s,t)\rangle = \mathbb{W} |\hat{P}(s,t)\rangle - s\, \hat{a}_2\, |\hat{P}(s,t)\rangle$$

diagonal operator of elements $a_2(\mathcal{C})$

is an example of <u>FEYNMAN-KAC</u> formula

jumps are described by the non-diagonal part of $\mathbb{W}$

. Second derivation of the result:

One has $A_2 = \sum_{k=1}^{K} (t_k - t_{k-1})\, a_2(\mathcal{C}_{k-1}) + (t - t_K) a_2(\mathcal{C}_K)$

Thus, using the expression previously obtained for the probability of an history.

$$\langle e^{-sA_2} \rangle = \sum_{K \geq 0} \sum_{\mathcal{C}_1 \ldots \mathcal{C}_K} \int_0^t dt_1 \ldots \int_{t_{K-1}}^t dt_K \; e^{-\sum_{k=1}^{K} (t_k - t_{k-1})\left(r(\mathcal{C}_{k-1}) + s a_2(\mathcal{C}_{k-1})\right) - (t - t_K)\left(r(\mathcal{C}_K) + s a_2(\mathcal{C}_K)\right)} \times \prod_{k=1}^{K} W(\mathcal{C}_{k-1} \to \mathcal{C}_k) \quad P_0(\mathcal{C}_0)$$

Indeed the derivation of the time-ordered exp. formula does not involve the conservation of probability presented by $\mathbb{W}$. It thus also works for $\mathbb{W}$.

And one recognizes precisely the expression of:

$$\langle - | \, T\exp \underbrace{\mathbb{W}_s(t)} \, | P_0 \rangle$$

time dependent operator of evolution corresponding precisely to $\mathbb{W}(s)$ for s associated to A_2

which is compatible with $|\hat{P}(s,t)\rangle = T\exp \mathbb{W}_s(t) \, |P_0\rangle$

. Remark: mixed observables:

The l.d.f. $\psi(s_1, s_2) = \lim_{t \to \infty} \frac{1}{t} \log \langle e^{-s_1 A_1 - s_2 A_2} \rangle$ is the max e.v. of

$$\boxed{\left(\mathbb{W}(s_1, s_2)\right)_{\mathcal{C}\mathcal{C}'} = \underbrace{e^{-s_1 a_1(\mathcal{C}' \to \mathcal{C})} W(\mathcal{C}' \to \mathcal{C}) - \left(r(\mathcal{C}) + s_2 a_2(\mathcal{C})\right)\delta_{\mathcal{C}\mathcal{C}'}}}$$

both diagonal and non-diagonal parts of the operator of evolution are modified, respectively by s_2 and s_1

Lecture on Stoch. processes — 2.13

Example:

- Time reversal symmetry and fluctuation theorem:

[J. Stat. Phys. **95** 333 (1999), Lebowitz & Spohn]

Consider the observable $Q_{\ell s} = \log \dfrac{W(\mathcal{C}_0\to\mathcal{C}_1)\cdots W(\mathcal{C}_{K-1}\to\mathcal{C}_K)}{W(\mathcal{C}_K\to\mathcal{C}_{K-1})\cdots W(\mathcal{C}_1\to\mathcal{C}_0)}$ ("entropy current")

Upon a jump $\mathcal{C}\to\mathcal{C}'$ one has $Q_{\ell s} \mapsto Q_{\ell s} + \log \dfrac{W(\mathcal{C}\to\mathcal{C}')}{W(\mathcal{C}'\to\mathcal{C})}$

The modified operator of evolution for s associated to $Q_{\ell s}$ is

$$\big(\mathbb{W}(s)\big)_{\mathcal{C}\mathcal{C}'} = e^{-s\log\frac{W(\mathcal{C}'\to\mathcal{C})}{W(\mathcal{C}\to\mathcal{C}')}}\, W(\mathcal{C}'\to\mathcal{C}) - r(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}$$

$$= W(\mathcal{C}'\to\mathcal{C})^{1-s}\, W(\mathcal{C}\to\mathcal{C}')^{s} - r(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}$$

Results of this class are among the few results valid in non-equilibrium steady states.

One thus has $\big(\mathbb{W}(s)\big)_{\mathcal{C}\mathcal{C}'} = \big(\mathbb{W}(1-s)\big)_{\mathcal{C}'\mathcal{C}}$

$$\mathbb{W}(s)^T = \mathbb{W}(1-s)$$

Relates rare events ($s=1$) to not rare ones ($s=0$). Implies Fluctuation-Dissipation Relation, and Onsager Relations.

Since these two operators have the same spectrum, one has $\boxed{\psi(s) = \psi(1-s)}$

This symmetry is an instance of 'Gallavotti-Cohen relation' [J. Stat. Phys **80**: 931 (1995)]

(or 'Fluctuation Theorem')

- Numerical algorithm for large deviation function:

s conjugated to $A = A_t$

By writing $W_s(\mathcal{C}\to\mathcal{C}') = e^{-s\,a(\mathcal{C}\to\mathcal{C}')}\,W(\mathcal{C}\to\mathcal{C}')$; $r_s(\mathcal{C}) = \sum_{\mathcal{C}'} W_s(\mathcal{C}\to\mathcal{C}')$

$$\big(\mathbb{W}(s)\big)_{\mathcal{C}\mathcal{C}'} = \underbrace{W_s(\mathcal{C}'\to\mathcal{C}) - r_s(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}}_{\text{(probability-conserving) evolution with modified rates } W_s(\mathcal{C}\to\mathcal{C}')} + \underbrace{\big(r_s(\mathcal{C}) - r(\mathcal{C})\big)\,\delta_{\mathcal{C}\mathcal{C}'}}_{\text{cloning with rate } r_s(\mathcal{C}) - r(\mathcal{C})}$$

for a population of <u>copies</u> of the system.

One can devise an algorithm to measure $\psi(s)$ numerically:

- evolution with rate $W_s(\mathcal{C}\to\mathcal{C}')$ of each copy
- cloning/pruning of copies with rate $r_s(\mathcal{C}) - r(\mathcal{C})$

favoring configurations with an atypical value of A.

then the global increase/decrease rate of the population is $\psi(s)$

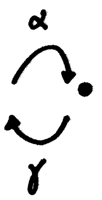
[There are tricks to keep the population size constant – see Refs.]

- **An example:** death and birth process on one site

$n=0$ site empty $\quad W(0\to 1) = \alpha$ (birth)

$n=1$ site occupied $\quad W(1\to 0) = \gamma$ (death)

(Reservoir)



$$\partial_t P(0,t) = \gamma\, P(1,t) - \alpha\, P(0,t)$$
$$\partial_t P(1,t) = \alpha\, P(0,t) - \gamma\, P(1,t)$$

$\alpha > 0$

$\gamma > 0$

$$\partial_t \underbrace{\begin{pmatrix} P(0,t) \\ P(1,t) \end{pmatrix}}_{|P(t)\rangle} = \underbrace{\begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}}_{\mathbb{W}} \underbrace{\begin{pmatrix} P(0,t) \\ P(1,t) \end{pmatrix}}_{|P(t)\rangle}$$

indeed

$\mathbb{W}_{01} = W(1\to 0) = \gamma$

$\mathbb{W}_{10} = W(0\to 1) = \alpha$

$\mathbb{W}_{00} = -r(0) = -W(0\to 1) = -\alpha$

$\mathbb{W}_{11} = -r(1) = -W(1\to 0) = -\gamma$

× search for a steady-state: one even has an equilibrium state (detailed balance ok)

"Grand-canonical" form $\quad P(n) \overset{?}{=} \frac{1}{Z} e^{\mu n} \quad$ μ = "chemical potential"

St.st.eq:

$$\left.\begin{aligned} 0 &= \gamma e^{\mu} - \alpha \\ 0 &= \alpha - \gamma e^{\mu} \end{aligned}\right\} \quad e^{\mu} = \frac{\alpha}{\gamma}$$

$$Z = e^{0\mu} + e^{\mu} = 1 + \frac{\alpha}{\gamma}$$

$$\boxed{P_{eq}(n) = \frac{\left(\frac{\alpha}{\gamma}\right)^n}{1 + \frac{\alpha}{\gamma}}} \quad \text{(normalized)}$$

In vector form: $|P_{eq}\rangle = \frac{1}{\alpha+\gamma}\begin{pmatrix} \gamma \\ \alpha \end{pmatrix}$ is indeed a right eigen-vector of $\mathbb{W}$ of e.v. 0.

In details: one rewrites

$$\partial_t |P(t)\rangle = (\alpha+\gamma)\begin{pmatrix} -\frac{\alpha}{\alpha+\gamma} & \frac{\gamma}{\alpha+\gamma} \\ \frac{\alpha}{\alpha+\gamma} & -\frac{\gamma}{\alpha+\gamma} \end{pmatrix} |P(t)\rangle$$

and one changes time so as to absorb $(\alpha+\gamma)$

× mean occupation number:

$$\boxed{\langle n\rangle = \sum_n n\, P_{eq}(n) = \frac{\alpha/\gamma}{1+\alpha/\gamma} = \frac{\alpha}{\alpha+\gamma}}$$

× reparametrization: one sets $\boxed{\alpha = c \mid \gamma = 1-c}$ which fixes the unit of time

$0<c<1$

Then:

$$\boxed{\begin{aligned} P_{eq}(n) &= c^n (1-c)^{1-n} \\ \langle n\rangle &= c \end{aligned}}$$

Binomial law

- In vector form $|P_{eq}\rangle = \begin{pmatrix} 1-c \\ c \end{pmatrix} = (1-c)|0\rangle + c\,|1\rangle$

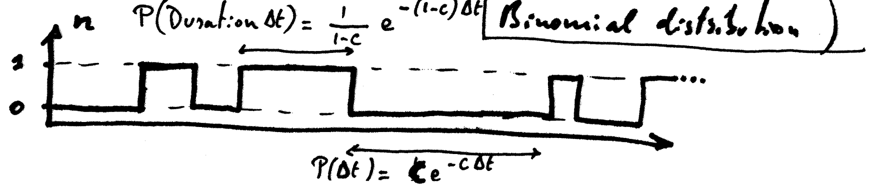
$$\mathbb{W} = \begin{pmatrix} c & 1-c \\ c & -(1-c) \end{pmatrix}$$

× generating function of the cumulant of the occupation number: $\langle e^{-sn}\rangle_{eq} = e^{\psi_{occ.}(s)}$

$$\psi_{occ.}(s) \equiv \log\langle e^{-sn}\rangle_{eq} = \log\left\{1-c + c\, e^{-s}\right\}$$

Describes all the statistics of the occupation number (which is trivial here: Binomial distribution)

× remark: typical history:



× <u>Large deviations of the time-averaged density</u> $\rho(t) = \int_0^t d\tau\, n(\tau)$:

This is an observable of type 2 (time-integrated)

<u>Mean value</u>: one expects $\langle \rho \rangle = \int_0^t d\tau \underbrace{\langle n(\tau) \rangle}_{\simeq\, c \text{ at large times}} \underset{t\to\infty}{\sim} ct$

<u>Variance</u>: $\langle \rho(t)^2 \rangle = \left\langle \left(\int_0^t d\tau\, n(\tau) \right)^2 \right\rangle = \int_0^t d\tau_1 \int_0^t d\tau_2 \underbrace{\langle n(\tau_1) n(\tau_2) \rangle}_{=?}$

difficult to determine since we don't know the 2-time correlation function

We thus go to the <u>generating functional of the cumulants</u> of ρ → c.g.f.

$$\langle e^{-s\rho(t)} \rangle \underset{t\to\infty}{\sim} e^{t\,\psi_\rho(s)}$$

The index ρ is there to remind us of which observable $\psi_\rho(s)$ is the c.g.f.

We have seen that

$$\boxed{\psi_\rho(s) = \operatorname{Max} \underset{\text{spectrum}}{\mathrm{Sp}}\, \mathbb{W}_\rho(s)} \quad , \quad \left(\mathbb{W}_\rho(s)\right)_{\mathcal{C}\mathcal{C}'} = W(\mathcal{C}' \to \mathcal{C}) - \big(r(\mathcal{C}) + s\, n(\mathcal{C})\big)\delta_{\mathcal{C}\mathcal{C}'}$$

$\rho(t) = \int_0^t d\tau\, n(\tau)$

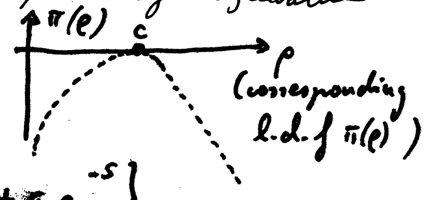
In our case:

$$\boxed{\mathbb{W}_\rho(s) = \begin{pmatrix} c & 1-c \\ c & -[(1-c)+s] \end{pmatrix}}$$

Eigenvalues are solutions in λ of $\det \begin{pmatrix} c-\lambda & 1-c \\ c & -\lambda-[(1-c)+s] \end{pmatrix} = 0$

One finds $\mathrm{Sp}\, \mathbb{W}_\rho(s) = \left\{ -\frac{1+s}{2} - \frac{1}{2}\sqrt{(1+s)^2 - 4cs} \,,\, \underbrace{-\frac{1+s}{2} + \frac{1}{2}\sqrt{(1+s)^2 - 4cs}}_{\text{always the largest eigenvalue}} \right\}$

$$\boxed{\psi_\rho(s) = -\frac{1+s}{2} + \frac{1}{2}\sqrt{(1+s)^2 - 4cs}} \quad \longrightarrow$$



which is very different from $\psi_{occ.}(s) = \log\{1 - c + c\, e^{-s}\}$

One can compare the successive cumulants from $(-1)^k \langle n^k \rangle_c = \frac{\partial^k}{\partial s^k}\Big|_{s=0} \psi_{occ.}(s)$; $(-1)^k \frac{1}{t} \langle \rho(t)^k \rangle_c = \frac{\partial^k}{\partial s^k}\Big|_{s=0} \psi_\rho(s)$

$\langle n \rangle = c$	$\frac{1}{t}\langle \rho \rangle = c$
$\langle n^2 \rangle_c = c(1-c)$	$\frac{1}{t}\langle \rho^2 \rangle_c = 2c(1-c)$
$\langle n^3 \rangle_c = c(1-c)(1-2c)$	$\frac{1}{t}\langle \rho^3 \rangle_c = 6c(1-c)(1-2c)$
$\langle n^4 \rangle_c = c(1-c)(1-6c(1-c))$	$\frac{1}{t}\langle \rho^4 \rangle_c = 24c(1-c)(1-5c(1-c))$
...	...

lecture on Stoch. processes

× Large deviations of the activity $K = \#\{\text{configuration changes}\}$:

. The mean value only depends on the equilibrium state, in the large time limit :

Indeed :

$$\partial_t P(n,K,t) = c\,P(0,K-1,t)\delta_{n1} + (1-c)P(1,K-1,t)\delta_{n0} - [c\,P(0,K,t) + (1-c)P(1,K,t)]$$

multiply by K and sum over K and n

$$\underbrace{\partial_t \sum_{n,K} K P(n,K,t)}_{\text{by definit}^{\circ}:\ \langle K\rangle} = \sum_K \Big\{ \overbrace{cKP(0,K-1,t) + K(1-c)P(1,K-1,t)}^{\text{for these terms of the sum, one does } K \to K+1} - cKP(0,K,t) - (1-c)KP(1,K,t)$$

$\langle K\rangle$: averages over histories on $[0,t]$

$$\partial_t \langle K\rangle = \sum_K \Big\{ c\,[\underbrace{K+1-K}_{=1}]\,P(0,K,t) + (1-c)[\underbrace{K+1-K}_{=1}]\,P(1,K,t)\Big\}$$

$$\partial_t \langle K\rangle = c\,P(0,t) + (1-c)P(1,t) \qquad \begin{cases} \sum_K P(0,K,t) = P(0,t) \\ \sum_K P(1,K,t) = P(1,t) \end{cases}$$

for $t\to\infty$:
$$\boxed{\underbrace{\partial_t \langle K\rangle}_{\text{average over histories}} = \underbrace{\langle c(1-n) + (1-c)\,n\rangle_{eq}}_{\text{average in the equilibrium steady state}} = \langle \underbrace{r(n)}\rangle_{eq}}$$

this relation is generic Check it!

$r(n) = c(1-n) + (1-c)\,n$ is the escape rate from configuration n.

In our case : $\boxed{\frac{1}{t}\langle K\rangle = 2c(1-c)}_{t\to\infty}$.

There is no generic way to extend this result for higher cumulants $\frac{1}{t}\langle K^k\rangle_c$ $k\geq 2$

. Generating function : $\psi_K(s) = \frac{1}{t}\log\langle e^{-sK}\rangle = \mathrm{Max}\ \mathrm{Sp}\ \mathbb{W}_K(s)$

$$\big(\mathbb{W}_K(s)\big)_{\mathcal{C}\mathcal{C}'} = e^{-s}W(\mathcal{C}'\to\mathcal{C}) - r(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'}\ ,\ \text{hence}\ \boxed{\mathbb{W}_K(s) = \begin{pmatrix} -c & e^{-s}(1-c) \\ e^{-s}c & -(1-c)\end{pmatrix}}$$

One finds $\boxed{\psi_K(s) = -\frac{1}{2} + \frac{1}{2}\sqrt{1-4c(1-c)(1-e^{-2s})}}$

No surprise here : everything is analytic

from which one gets for instance

$$\frac{1}{t}\langle K\rangle = 2c(1-c)$$

$$(t\to\infty)\qquad \frac{1}{t}\langle K^2\rangle_c = 4c(1-c)(1-2c(1-c))$$

$$\vdots$$

Population algorithm to measure $\psi_K(s)$: for $s \leftrightarrow K$

$$\partial_t \hat{P}(n,s,t) = \underbrace{e^{-s}\left[cn + (1-c)(1-n)\right]\hat{P}(1-n,s,t) - e^{-s}\left(c(1-n) + n(1-c)\right)\hat{P}(n,s,t)}_{\text{death \& birth process with rates } \circ \xrightarrow{e^{-s}c} \bullet,\ \bullet \xrightarrow{e^{-s}(1-c)} \circ}$$

$$+ \underbrace{\left(e^{-s}-1\right)\big(\underbrace{c(1-n) + n(1-c)}_{r(n)}\big)\hat{P}(n,s,t)}_{\text{cloning at rate } (e^{-s}-1)\, r(n)}$$

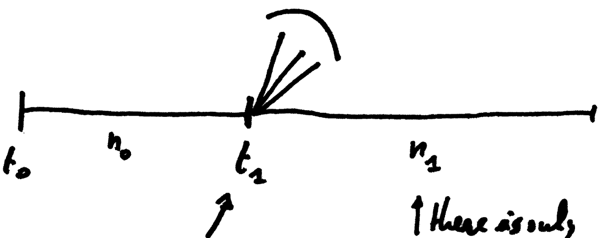
For this simple example, we keep total population $\mathcal{N}(t)$ non-constant.

Start with $\mathcal{N}_0$ copies of the system ($\mathcal{N}_0 \gg 1$, typically $10^2 - 10^3$)

Take $s < 0$ (so that $e^{-s} - 1 > 1$).

Each copy of the system evolves as follow:

gives rise to p offsprings: $p+1 = \underbrace{\lfloor (e^{-s}-1)\, r(n_0) \rfloor}_{\text{this is}}$ ← lower integer part



initial state (e.g. $n=0$) t_0 — n_0 — t_1 — n_1 and so on

$P(t_1) = r_s(n_0)\, e^{-t_1 / r_s(n_0)}$

there is only one choice for n_1: $1-n_0$

Remark: it is better to take $p+1 = \lfloor (e^{-s}-1)\, r(n_0) + \varepsilon \rfloor$ where ε is uniformly distributed on $[0,1[$

Each of the offsprings evolves independently afterwards.

In the large time limit: $\mathcal{N}(t) \sim e^{t\,\psi_K(s)}$