Lecture 3: From operators to field theories

• MOTIVATIONS:

× Quantum Mechanics: the wave function of being in x at time t, having started from a state Ψ_0 at time 0 is

$$\Psi(x,t) = \langle x | e^{i t \hbar H} | \Psi_0 \rangle$$ where H is the Schrödinger operator.

Feynman's approach to compute $\Psi(x,t)$ is to rewrite this propagator-like expression as a integral over all the possible paths followed by the particle:

$$\Psi(x,t) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x(\tau)\, e^{i \hbar \underbrace{S[x(\tau),t]}_{\text{action whose expression is deduced from } H}}$$

There are many ways to construct such a "path integral" form. One method which is particularly well adapted is the use of coherent states

× Stochastic processes: One has $|P(t)\rangle = e^{tW} |P_0\rangle$

and thus the mean value of an observable O depending on the final state $|P(t)\rangle$ is:

$$\langle O(t) \rangle = \langle - | O\, e^{tW} \underbrace{|P_0\rangle}_{\text{initial state}}$$

× In a way very similar to that of the quantum mechanics, one will write $\langle O(t) \rangle$ in terms of a path-integral form.

(Bosonic) coherent states

- **Remark**: the equilibrium solution of the birth & death process
  $$A \underset{c}{\overset{1}{\rightleftarrows}} 0 \quad \text{of operator} \quad \mathbb{W} = ca + ca^{+} - c\hat{n} - c$$
  is the Poisson law of density c: $P_{eq}(n) = e^{-c}\frac{c^n}{n!}$

- This is easily checked from the detailed balance on rates $\begin{cases} W(n\to n+1) = c \\ W(n \to n-1) = n \end{cases}$

- One can also check that $\mathbb{W}|P_{eq}\rangle = 0$:

  Indeed: $a|P_{eq}\rangle = e^{-c}\sum\limits_{n\geq 0}\frac{c^n}{n!}n|n-1\rangle = e^{-c}c\sum\limits_{n\geq 0}\frac{c^{n-1}}{(n-1)!}|n-1\rangle \overset{n\to n+1}{=} e^{-c}c\sum\limits_{n\geq 0}\frac{c^n}{n!}|n\rangle$

  hence: $\boxed{a|P_{eq}\rangle = c|P_{eq}\rangle}$

  and $a^{+}|P_{eq}\rangle = e^{-c}\sum\limits_{n\geq 0}\frac{c^n}{n!}|n+1\rangle \overset{n\to n-1}{=} \frac{1}{c}e^{-c}\sum\limits_{n\geq 0}\frac{c^n}{(n-1)!}|n\rangle = \frac{\hat{n}}{c}|P_{eq}\rangle$ $\quad (n-1)! = \frac{1}{n}n!$

  $$\boxed{a^{+}|P_{eq}\rangle = \frac{\hat{n}}{c}|P_{eq}\rangle}$$

- Finally: $\mathbb{W}|P_{eq}\rangle = (a + ca^{+} - \hat{n} - c)|P_{eq}\rangle = \left(c + c\frac{\hat{n}}{c} - \hat{n} - c\right)|P_{eq}\rangle = 0$

- **Coherent state**: one has thus seen that $a|P_{eq}\rangle = c|P_{eq}\rangle$: $|P_{eq}\rangle$ is a right eigenvector of the annihilation operator a.
  This is a coherent state.

- **Left eigenvector of a^{+}**:
  - action of a^{+} on $\langle n|$: $\langle n|a^{+} = \sum\limits_m \langle n|a^{+}|m\rangle\langle m| = \langle n-1|$: $\boxed{\langle n|a^{+} = \langle n-1|}$ with $\langle 0|a^{+} = 0$

    $\underbrace{\langle n|a^{+}|m\rangle}_{\delta_{n,m+1}} \Rightarrow m = n-1$

  - Let's now compute $\langle P_{eq}|a^{+}$

    $\langle P_{eq}|a^{+} = e^{-c}\sum\limits_{n\geq 0}\langle n-1|\frac{c^n}{n!} = e^{-c}\sum\limits_{n\geq 0}\langle n|\frac{c^n}{n!}\frac{1}{n+1}$ → this is not a good eigenvector.

- **Left and right coherent states**: $z \in \mathbb{C}$

$$\boxed{\begin{aligned} |z\rangle &= e^{-\frac{1}{2}|z|^2}\sum_{n\geq 0}\frac{z^n}{n!}|n\rangle \\ \langle z| &= e^{-\frac{1}{2}|z|^2}\sum_{n\geq 0}(z^*)^n\langle n| \end{aligned}}$$

one has $a|z\rangle = z|z\rangle$

$\langle z|a^{+} = z^*\langle z|$

• Properties:

× normalisation: $\langle z|z\rangle = e^{-|z|^2}\sum_{n,m}\frac{z^{*n}z^{m}}{m!}\overbrace{\langle n|m\rangle}^{\delta_{nm}} = 1$ $\qquad \boxed{\langle z|z\rangle = 1}$

× scalar product: $\langle z_1|z_2\rangle = e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2}\sum_{n\geq 0}\frac{(z_1^* z_2)^n}{n!} = e^{-\frac{1}{2}\underbrace{(|z_1|^2+|z_2|^2)}_{|z_1-z_2|^2+z_1^*z_2+z_1z_2^*}+z_1^*z_2}$

$$\boxed{\langle z_1|z_2\rangle = \exp\Big(-\tfrac{1}{2}|z_1-z_2|^2+\tfrac{1}{2}\underbrace{(z_1^*z_2-z_1z_2^*)}_{2\,\mathrm{Im}\,z_1^*z_2}\Big)}$$

× representat° of the identity: (central formula for the construction of path $\int$)

. lemma: $\int_{\mathbb{C}}\frac{dx\,dy}{\pi} z^{*n}z^{m}e^{-\frac{1}{2}|z|^2} = n!\,\delta_{nm}$ $\qquad \int = \int_0^{2\pi}d\theta\cdot\int_0^{+\infty}dr\,r\,r^{2n}e^{-\frac{1}{2}r^2} = \frac{1}{2}n!\,2\pi$

δ is obvious by angular integration

. hence: $\int_{\mathbb{C}}\frac{dx\,dy}{\pi}|z\rangle\langle z| = \int_{\mathbb{C}}\frac{dx\,dy}{\pi}\sum_{n,m}\frac{(z^*)^n z^m}{m!}e^{-|z|^2}|n\rangle\langle m| = \sum_{n\geq 0}\frac{n!}{n!}|n\rangle\langle n|$

$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}|z\rangle\langle z| = \mathbb{1}}$$

$\mathbb{1} = \sum_n |n\rangle\langle n|$ is the identity in the Fock space ($|n\rangle$)

× representation of the Fock vectors:

$$\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}e^{-\frac{1}{2}|z|^2}|z\rangle = \sum_{m\geq 0}\overbrace{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}z^{m}\frac{e^{-|z|^2}}{m!}}^{\frac{n!}{m!}\delta_{nm}=\delta_{nm}}|m\rangle = |n\rangle$$

thus

$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}e^{-\frac{1}{2}|z|^2}|z\rangle = |n\rangle}$$

and similarly

$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}\frac{z^{n}}{n!}e^{-\frac{1}{2}|z|^2}\langle z| = \langle n|}$$

Lecture on Stoch. processes

- **Construction of the path integral:** ($\vec{n}$ may be a vector $\vec{n}$)

× Let's consider a system starting from distribution P_0 at time 0 and evolving with operator $\mathbb{W}$ (or $\mathbb{W}(s)$ when considering l.d.f.)

The average of an observable $O(n)$ writes, at time t: $\langle O\rangle = \sum_n O(n) P(n,t)$

$$\langle O\rangle = \langle -|O(\hat{n})|P(t)\rangle = \langle -|O(\hat{n})\, e^{t\mathbb{W}}|P_0\rangle$$

× Let's decompose $[0,t]$ into $\mathcal{N}$ steps $dt = \frac{t}{\mathcal{N}}$ and insert $\mathcal{N}+1$ representations of the identity $\mathbb{1} = \int_{\mathbb{C}} \frac{d^2 z_p}{\pi} |z_p\rangle\langle z_p|$ $\quad \left(d^2 z_p \equiv d\,\mathrm{Re}\, z_p\, d\,\mathrm{Im}\, z_p\right)_{0\le p\le \mathcal{N}}$

in the (exact relation) $e^{t\mathbb{W}} = e^{\mathcal{N} dt\, \mathbb{W}} = e^{dt\mathbb{W}} \cdot \ldots \cdot e^{dt\mathbb{W}}$ ($\mathcal{N}$ factors)

$$\langle O\rangle = \langle -|O(\hat{n})\overset{\mathbb{1}}{\downarrow} e^{dt\mathbb{W}} \overset{\mathbb{1}}{\downarrow}\cdot \ldots \cdot \overset{\mathbb{1}}{\downarrow} e^{dt\mathbb{W}} \overset{\mathbb{1}}{\downarrow}|P_0\rangle$$

$$= \int_{\mathbb{C}^{\mathcal{N}+1}} \frac{d^2 z_0}{\pi}\cdots\frac{d^2 z_{\mathcal{N}}}{\pi} \langle -|O(\hat{n})|z_{\mathcal{N}}\rangle\langle z_{\mathcal{N}}|e^{dt\mathbb{W}}|z_{\mathcal{N}-1}\rangle \cdots\cdots \langle z_1|e^{dt\mathbb{W}}|z_0\rangle\langle z_0|P_0\rangle$$

× One has changed the product of $\mathcal{N}$ operators into a product of $\mathcal{N}$ numbers $\langle z_p|e^{dt\mathbb{W}}|z_{p-1}\rangle$

× **Bulk terms**: let's "normal order" $\mathbb{W}$ (all $a^\dagger$'s on the left, all a's on the right) using the commutation relation on the a & $a^\dagger$'s.

One defines the $\mathbb{C}$-valued function $W(z_2^*, z_1)$ as $\mathbb{W}$ where $\begin{cases} a^\dagger \text{ is replaced by } z_2^* \\ a \text{ is replaced by } z_1\end{cases}$

Then: in the limit $dt\to 0$: $\left[\begin{array}{l}\text{using that } |z\rangle (\langle z|) \text{ is a} \\ \text{right (left) ev of } a\ (a^\dagger)\end{array}\right]$

$$\langle z_{p+1}|e^{dt\mathbb{W}}|z_p\rangle = \langle z_{p+1}|1 + dt\,\mathbb{W}|z_p\rangle = \langle z_{p+1}|z_p\rangle + dt\underbrace{\langle z_{p+1}|\mathbb{W}|z_p\rangle}_{W(z_{p+1}^*,\, z_p)} + O(dt^2)$$

$$= \langle z_{p+1}|z_p\rangle\left(1 + dt\,\frac{W(z_{p+1}^*, z_p)}{\langle z_{p+1}|z_p\rangle}\right) + O(dt^2)$$

$$\boxed{\langle z_{p+1}|e^{dt\mathbb{W}}|z_p\rangle = \langle z_{p+1}|z_p\rangle \exp\left(dt\,\frac{W(z_{p+1}^*, z_p)}{\langle z_{p+1}|z_p\rangle}\right) + O(dt^2)}$$

× **Boundary terms**: One defines $O(z_{\mathcal{N}}) = \dfrac{\langle -|O(\hat{n})|z_{\mathcal{N}}\rangle}{\langle -|}$, with $\langle -|a^\dagger = \langle -|$; $O(z_{\mathcal{N}}) = O(\hat{n})$ with $\begin{cases} a^\dagger \text{ to } 1 \\ a \text{ to } z_{\mathcal{N}}\end{cases}$

and $P(z_0) = \langle z_0|P_0\rangle$

Besides: $\langle -| = e^{\frac{1}{2}}\langle \underline{1}|$ (coherent state) and thus $\langle -|O(\hat{n})|z_{\mathcal{N}}\rangle = e^{\frac{1}{2}} e^{-\frac{1}{2}-\frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}} O(z_{\mathcal{N}}) = e^{-\frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}} O(z_{\mathcal{N}})$

- Continuous time limit $\mathcal{N}\to\infty$ (ie $dt\to 0$):

One assumes that, in the integral, the values of z_k's which dominate are such that

$$z_{p+1} - z_p = \mathcal{O}(dt)$$

. Then, if W is regular enough (it is a polynomial in general)

$$\frac{W(z^*_{p+1}, z_p)}{\langle z_{p+1}|z_p\rangle} = W(z^*_p, z_p) + \mathcal{O}(dt)$$

. Besides, one wants to write $\langle z_{p+1}|z_p\rangle$ as $e^{dt\ldots}$, or, better, exactly:

$$\langle z_{p+1}|z_p\rangle = \exp\Big(-\frac{1}{2}\underbrace{|z_{p+1}-z_p|^2}_{(z^*_{p+1}-z^*_p)(z_{p+1}-z_p)} + \frac{1}{2}z^*_{p+1}z_p - \frac{1}{2}z^*_p z_{p+1}\Big)$$

$$= \exp\Big(+\frac{1}{2}z^*_{p+1}z_{p+1} - \frac{1}{2}z^*_p z_p + z^*_{p+1}z_p - z^*_{p+1}z_{p+1}\Big)$$

$$\boxed{\langle z_{p+1}|z_p\rangle = \exp\Big[\underbrace{\tfrac{1}{2}\big(|z_{p+1}|^2 - |z_p|^2\big)}_{\text{useful when summing: this becomes a telescopic sum.}} - z^*_{p+1}(z_{p+1}-z_p)\Big]}$$

- Gathering all steps:

$$\langle \mathcal{O}\rangle = \int \frac{d^2z_0}{\pi}\cdots\frac{d^2z_{\mathcal{N}}}{\pi}\exp\Bigg(\underbrace{\overbrace{\sum_{p=0}^{\mathcal{N}-1}\Big[\tfrac{1}{2}|z_{p+1}|^2 - \tfrac{1}{2}|z_p|^2 - z^*_{p+1}(z_{p+1}-z_p)}^{\text{exact}}}_{\frac{1}{2}|z_{\mathcal{N}}|^2 - \frac{1}{2}|z_0|^2} + \overbrace{dt\,W(z^*_p, z_p) + \mathcal{O}(dt^2)}^{\text{approximate (but exact as } dt\to 0)}\Big]\Bigg)$$

$$\times P(z_0)\,\mathcal{O}(z_{\mathcal{N}})$$
$$\times e^{-\frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}}$$

. Finally: denoting $\begin{array}{l} z_p = \varphi(p\,dt) \\ z^*_p = \bar\varphi(p\,dt)\end{array}$, $\int \prod_k \frac{d^2 z_k}{\pi} = \int \mathcal{D}\varphi\,\mathcal{D}\bar\varphi$, $z_{p+1}-z_p = dt\,\partial_t\varphi$, $\sum dt = \int dt$

One finds the 'path integral expression':

$$\boxed{\begin{aligned}\langle\mathcal{O}\rangle &= \int\mathcal{D}\varphi\,\mathcal{D}\bar\varphi\;\mathcal{O}(\varphi(t))\,P_0(\varphi(0))\,e^{-S[\hat\varphi,\hat\varphi]}\\ S[\hat\varphi,\hat\varphi] &= -\varphi(t) + \tfrac{1}{2}\varphi^2(0) + \int_0^t d\tau\,\big[\bar\varphi\,\partial_\tau\varphi - W(\bar\varphi,\varphi)\big]\end{aligned}}$$

- Example to check that everything works: $A \underset{c}{\overset{1}{\rightleftarrows}} 0$ — Lecture on stoch. processes (s.o)

* Let's determine the mean value of n at final time, with s associated to the activity K

One has $\partial_t P(n,s,t) = e^{-s}\left[cP(n-1,s,t) + P(n+1,s,t)\right] - (n+c)P(n,s,t)$

ordinarily $\boxed{\mathbb{W}(s) = e^{-s}(ca^\dagger + a) - (c + \hat{n})}$

ρ may depend on s

Let's search an eigen state of the Poisson form: $|P(s)\rangle = e^{-\rho(s)}\sum_{n\geq 0} \frac{\rho(s)^n}{n!}|n\rangle$

One has: $\mathbb{W}(s)|P(s)\rangle = \left(e^{-s}\left(\frac{c}{\rho}\hat{n} + \rho\right) - (c + \hat{n})\right)|P(s)\rangle$ cf: $\begin{cases} a|\rho\rangle = \rho|P\rangle \\ a^\dagger|\rho\rangle = \frac{1}{\rho}\hat{n}|P\rangle \end{cases}$

$\left(e^{-s}\frac{c}{\rho} - 1\right)\hat{n} = 0$ if $\boxed{\rho = ce^{-s}}$

in that case

$= c\left(e^{-2s} - 1\right)P(s)$

In other words, [since $\psi(0) = 0$, this is indeed the max eigenv.]

$\boxed{|P(s)\rangle \text{ Poissonian of density } \rho = ce^{-s} \text{ is the eigenvector corresponding to } \psi(s) = c\left(e^{-2s} - 1\right)}$

* In the path integral formalism: $\langle \hat{n} \rangle_s = \dfrac{\langle \hat{n} e^{-sK}\rangle}{\langle e^{-sK}\rangle}$ with $\langle \hat{n} e^{-sK}\rangle = \int \mathcal{D}\varphi\mathcal{D}\bar{\varphi} \ldots$

with $\mathcal{O}(\varphi) = \varphi$, $W_s(\bar{\varphi},\varphi) = e^{-s}(c\bar{\varphi} + \varphi) - (c + \bar{\varphi}\varphi)$

Steady-state saddle point equations $\frac{\delta S}{\delta \varphi} = 0 = \frac{\delta S}{\delta \bar{\varphi}}$ write:

$$\begin{cases} 0 = \dfrac{\partial W_s}{\partial \varphi} \\ 0 = \dfrac{\partial W}{\partial \bar{\varphi}} \end{cases} \quad \text{ie} \quad \begin{cases} 0 = e^{-s} - \bar{\varphi} \\ 0 = ce^{-s} - \varphi \end{cases} \quad \text{hence} \quad \boxed{\begin{cases} \bar{\varphi}_s = e^{-s} \\ \varphi_s = ce^{-s} \end{cases}}$$

the are steady-state saddle point solutions

Thus: $\psi = \frac{1}{t}S_{\star}[\varphi_s, \bar{\varphi}_s] = \frac{t}{t}W(\bar{\varphi}_s, \varphi_s) = e^{-s}\left(ce^{-s} + ce^{-s}\right) - \left(c + ce^{-2s}\right) = c\left(e^{-2s} - 1\right)$

$\underline{\psi(s) = c\left(e^{-2s} - 1\right) \text{ as expected}}$ and the mean density is $\varphi_s = ce^{-s} : \alpha$.

* Remark: for the density at intermediate time: $e^{\frac{t}{2}\mathbb{W}}\mathcal{O}(\hat{n})e^{\frac{t}{2}\mathbb{W}} \longleftrightarrow \langle z_p|\mathcal{O}|z_{p-1}\rangle$

One thus should replace $\mathcal{O}(z_N)$ by $\mathcal{O}^{\text{interm}}(\bar{z}_p, z_{p-1}) = \mathcal{O}^{\text{interm}}(\bar{z}_p, z_p) + \mathcal{O}(dt)$ with $\begin{cases} a \leadsto z_p \\ a^\dagger \leadsto \bar{z}_p \end{cases}$

In our case one finds $\boxed{\rho^{\text{interm}}(s) = \bar{\varphi}_s\varphi_s = ce^{-2s}}$ which is a less trivial result.

- Diffusion of particles; diffusive limit:

We consider a lattice of L sites with particles diffusing symmetrically at rate 1

$$\mathbb{W} = \sum_{k=1}^{L} a_k^\dagger a_{k+1} + a_k a_{k+1}^\dagger - \hat{n}_k - \hat{n}_{k+1} = \sum_{k=1}^{L} -\left(a^\dagger_{k+1} - a^\dagger_k\right)\left(a_{k+1} - a_k\right) \qquad \text{p.b.c.}\ L+1 \equiv 1$$

The construction is similar as previously, with coherent state on each site.

$$\boxed{S[\hat\varphi,\varphi] = \int_0^t d\tau \Big[\sum_k \big\{ \underbrace{(\hat\varphi_{k+1} - \hat\varphi_k)}_{\equiv \nabla_k \hat\varphi} \underbrace{(\varphi_{k+1} - \varphi_k)}_{\equiv \nabla_k \varphi} + \hat\varphi_k \partial_\tau \varphi_k \big\} \Big] + \sum_k \frac{1}{2} |\varphi_k^2(0) - \varphi_k(t)}$$

$$\int_0^t d\tau\, L^2 \underbrace{\int dx\, L}_{\sum} \; L^{-1}\partial_x \hat\varphi\, L^{-1} \partial_x \varphi \qquad \text{↳ } L\int \sum$$

Besides: $\sum_k \nabla_k \hat\varphi \nabla_k \varphi = - \sum_k \hat\varphi_k \nabla_k \nabla_{k} \varphi_k$ (t)

Continuous-space limit: $x = k\frac{1}{L}$, $\frac{1}{L} = a$: lattice step, $\nabla_k \varphi \leftrightarrow \frac{\partial_x \varphi}{a^{-1}} = L^{-1} \partial_x \varphi$

Diffusive scaling: the time that we have in (t) is microscopic: $t = t_{mic}$, $\tau = \tau_{mic}$

To go to the macroscopic scale, one sets: $t_{mic} = L^2 t$, $\tau_{mic} = L^2 \tau$ — $\boxed{\text{DIFFUSIVE SCALING}}$ — i.e.: $d\tau_{mic} = L^2 d\tau$, $\partial_{\tau_{mic}} = L^{-2} \partial_\tau$

Finally, in terms of the new field:

$$\boxed{S[\hat\varphi,\varphi] = L \int_0^t d\tau \Big[\int_0^1 dx \big\{ \hat\varphi (\partial_t - \partial_x^2) \varphi \big\} \Big] + \int dx \frac{1}{2} \hat\varphi(0)\varphi(0) - \varphi(t)}$$

in what follows one forgets about boundary terms

- Cole-Hopf transform: One would like $\rho = \hat\varphi \varphi$ to play a role. It happens a correct representation is $\boxed{\hat\varphi = e^{\hat\rho},\ \varphi = \rho e^{-\hat\rho}}$

$$\hat\varphi \partial_t \varphi = e^{\hat\rho} (\partial_t \rho - \rho \partial_t \hat\rho) e^{-\hat\rho} = \partial_t(\rho - \rho\hat\rho) + \hat\rho \partial_t \rho$$

$$\partial_x \hat\varphi \partial_x \varphi = e^{\hat\rho} (\partial_x \hat\rho)(\partial_x \rho - \rho \partial_x \hat\rho) e^{-\hat\rho} = \partial_x \hat\rho \partial_x \rho - \rho (\partial_x \hat\rho)^2.$$

Hence:

$$\boxed{S[\rho,\hat\rho] = L \int_0^t d\tau \int_0^1 dx \big\{ \hat\rho(\partial_t - \Delta)\rho + \rho(\partial_x \hat\rho)^2 \big\}} + \text{boundary terms.}$$

- Simple case to train : without space.

One consider a field $f(t)$ satisfying

$$\underbrace{\partial_t f}_{\equiv \frac{f(t+dt)-f(t)}{dt}} = \underbrace{-V'(f) + \eta(t)}_{\text{at time } t \text{ and not } t+dt}$$

$\eta(t)$ white noise with $\langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$

i.e. $P[\eta] \propto \exp\left(-\frac{1}{2}\underbrace{\int_0^t d\tau \, \frac{\eta^2(t)}{2}}_{\int_0^t\int_0^t d\tau d\tau' \frac{\delta(\tau-\tau')}{2} f(\tau) f(\tau')}\right)$

How can represent the probability of an history as $\langle \delta(\ldots)\rangle$ and hence:

$$P[\eta] \propto \int_{i\mathbb{R}} \mathcal{D}\hat{f} \exp\left(-\int_0^t \frac{d\tau}{2} \, 2\hat{f}^2(\tau) + \hat{f}(\tau)\eta(\tau)\right)$$

$$\overbrace{\langle O(t)\rangle}^{\text{over histories of duration } t} = \int_{0\le\tau\le t} \mathcal{D}f(\tau)\mathcal{D}\eta(\tau) \, \underbrace{\overbrace{\delta\left(\partial_t f + V'(f) - \eta(t)\right)}^{\text{imposes the equation of Langevin}}} P[\eta] \; P_0(f)$$

| the $\delta(\ldots)$ is a 'product' over each time step of Dirac delta's on $f(t)$.

→ | so as to integrate over η, one transform it into a product of Dirac delta's on η, step by step in time

⇒ | this induces a Jacobian, which is unity (or constant) thanks to the choice of Itô convention $\partial_t f \leadsto \frac{f(t+dt)-f(t)}{dt}$

Integrating now on η, one finds

$$\boxed{\langle O(t)\rangle = \int \mathcal{D}f\mathcal{D}\hat{f} \, \exp\left(-\int_0^t d\tau \left[\hat{f}\cdot(\partial_t f + V'(f)) + \hat{f}^2\right]\right) O(f(t)) \, P_0(f(0))}$$

• Martin-Siggia-Rose formalism of Langevin Dynamics

+ : Janssen, de Dominicis

$x \in \mathbb{R}^d$ position of particles.

$\rho(x,t)$ density of particles.

Again: Itô discretization

Langevin equation for $\rho(x,t)$:

$$\partial_t \rho = -\partial_x\left(-\partial_x \rho + \xi\right)$$

with $\xi(x,t)$ white noise of variance

small noise $\rightsquigarrow$ $\langle \xi(x,t)\,\xi(x',t')\rangle = \frac{2}{L}\rho(x,t)\,\delta(x-x')\,\delta(t-t')$

i.e.: $$P[\xi] \propto \exp\left[-\frac{1}{2}\int dx\,dt\, \frac{\xi^2(x,t)}{\frac{2}{L}\rho(x,t)}\right] \overset{\text{e.g.}}{\propto} \int_{i\mathbb{R}} \mathcal{D}\hat\rho\, \exp\left[-\int dx\,dt\, L\,\xi\,\partial_x\hat\rho + \rho\,(\partial_x\hat\rho)^2\right]$$

. Average of an observable:

$$\langle \mathcal{O}(t)\rangle = \int \mathcal{D}\rho\,\mathcal{D}\xi\;\; \delta\big(\partial_t\rho + \partial_x(-\partial_x\rho + \xi)\big)\, P[\xi]\;\; P_0(\rho(0))$$

integrating over ξ

$$\boxed{\begin{aligned}&\langle \mathcal{O}(t)\rangle = \int \mathcal{D}\rho\,\mathcal{D}\hat\rho\,\exp\left(-S[\hat\rho,\rho]\right) P_0(\rho(0)) \quad\text{, with the action}\\ &S[\hat\rho,\rho] = L\int_0^t d\tau \int dx\,\left(\hat\rho\,(\partial_t\rho - \partial_x^2\rho) + \rho\,(\partial_x\hat\rho)^2\right)\end{aligned}}$$

This is the same action as previously

It gives, doing the steps, a construction of the mesoscopic $\overbrace{\text{Langevin equation}}^{\text{otherwise, one has to come back to phenomenological Langevin equations.}}$

microscopic dynamics $\longrightarrow$ operator of evolution $\mathbb{W}$ [or $\mathbb{W}(s)$] $\longrightarrow$ Doi-Peliti construction $\langle\ldots\rangle = \int \mathcal{D}\rho\,\mathcal{D}\hat\rho\; e^{-S[\rho,\hat\rho]}$ in discrete space

$\big\downarrow$ $\rightarrow$ determination of the correct diffusive and scaling

$\langle\ldots\rangle = \int \mathcal{D}\rho\,\mathcal{D}\hat\rho\; e^{-S[\rho,\hat\rho]}$ in continuous space $\longrightarrow$ Reading as a (possible) Langevin dynamics

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Lecture on stoch. processes

• Counter-example:

$$A + A \xrightarrow{\gamma} 0$$

$aa^+ = a^+a + 1$

normal order

$$\mathbb{W} = \gamma\left(a^2 - \hat{n}(\hat{n}-1)\right)$$

$$a^+a(a^+a-1) = a^+(a^+a+1)a - a^+a = a^{+2}a^2$$

$$\boxed{\mathbb{W} = \gamma(a^{+2}-1)a^2}$$ thus

$$S[\hat\varphi,\varphi] = \int_0^t d\tau \left(\hat\varphi\partial_t\varphi - \gamma(\hat\varphi^2-1)\varphi^2\right) \quad + \text{ bound.}$$

One performs the similarity transformation $\hat\varphi \mapsto \hat\varphi - 1$

$$S[\hat\varphi, \varphi] = \int_0^t d\tau \left(\hat\varphi\partial_t\varphi - \gamma(\hat\varphi^2 - 2\hat\varphi)\varphi^2\right)$$

$$\hat\varphi(\partial_t + 2\gamma\varphi)\varphi \quad - \gamma\varphi^2\hat\varphi^2$$

$$\overset{?}{=} \int_0^t d\tau \; \hat\varphi(\partial_t + V'(\varphi))\varphi \underset{\uparrow}{+} \gamma \underbrace{\sigma(\varphi)}_{>0}\hat\varphi^2$$

the sign correspond to an "imaginary" noise

This means there is no Langevin (real noise) description of the reaction process $A + A \to 0$

The action is only well defined when one uses the

$$\mathbb{W} \rightsquigarrow S[\varphi\hat\varphi]$$ construction.

do: Peliti

• $SU(2)$ COHERENT STATES :

• **Definition** : by analogy, one choses a Bernoulli-like form :

$$\boxed{\begin{aligned} |z\rangle &= \frac{1}{(1+|z|^2)^{N/2}} \sum_{0\le n\le N} \binom{N}{n} z^n |n\rangle \\ \langle z| &= \frac{1}{(1+|z|^2)^{N/2}} \sum_{0\le n\le N} z^{*n} \langle n| \end{aligned}}$$

• **Properties** :

× normalization : $\boxed{\langle z|z\rangle = 1}$

× representation of the identity :

Non uniform measure

$$\boxed{\int_{\mathbb{C}} d\mu(z)\, |z\rangle\langle z| = \mathbb{1} \quad \text{with} \quad d\mu(z) = \frac{N+1}{\pi} \frac{d^2 z}{(1+|z|^2)^2}}$$

Exercise : check this representation using a determination of

$$\int_{\mathbb{C}} d\mu(z)\, \frac{z^n z^{*m}}{(1+|z|^2)^N}$$

× mean values of operators : $SU(2)$ coherent states are NOT eigenvectors

In the construction, in fact, one only needs

$$\boxed{\begin{aligned} \langle z|S^+|z\rangle &= N\frac{z^*}{1+|z|^2} \\ \langle z|S^-|z\rangle &= N\frac{z}{1+|z|^2} \\ \langle z|S^z|z\rangle &= \frac{1}{2}N\frac{|z|^2-1}{|z|^2+1} \end{aligned}}$$

or using

$$z = \frac{\rho}{1-\rho} e^{-\hat{\rho}} \qquad \bar{z} = e^{\hat{\rho}}$$

$$\boxed{\begin{aligned} \langle z|S^+|z\rangle &= (1-\rho)e^{\hat{\rho}}N \\ \langle z|S^-|z\rangle &= \rho\, e^{-\hat{\rho}}N \\ \langle z|S^z|z\rangle &= (2\rho-1)N \\ \text{ie} \quad \langle z|\hat{n}|z\rangle &= \rho N \end{aligned}}$$

* remark : since $|z\rangle$ is not an eigenvector of S^- one do not has $\langle z|S^-S^-|z\rangle = \left(N\frac{z}{1+|z|^2}\right)^2$ contrary to the case of Doi-Peliti where $a^2|z\rangle = z^2|z\rangle$.

There are know formula to express the mean value $\langle z|S^-S^-|z\rangle$ (This encodes the finiteness of N) (see references)

- Path integral representation :

Following the same path as for Bosonic coherent space one arrives at:

$$\boxed{\begin{aligned} S[\hat{\rho},\rho] &= N\int_0^t dt\left[\hat{\rho}\,\partial_t\rho - W(\hat{\rho},\rho)\right] \\ W(\hat{\rho},\rho) &= \frac{1}{N}\langle z|\mathbb{W}|z\rangle \end{aligned}}$$

computed by the replacement rules

$S^+ \leftrightarrow (1-\rho)e^{+\hat{\rho}}$
$S^- \leftrightarrow \rho\, e^{-\hat{\rho}}$
$\hat{n} \leftrightarrow \rho$

Boundary terms need special care.

The same formula applies with s-modified dynamics so as to represent large deviation functions.

- Including several lattice sites : and keeping space discrete

$$S[\hat{\rho},\rho] = N\left\{\sum_{k=1}^{L}\int_0^t d\tau\left[\hat{\rho}_k\partial_t\rho_k\right] + \int_0^t d\tau\, W(\vec{\hat{\rho}},\vec{\rho})\right\}$$

- ## Case of Simple exclusion processes : the symmetric one

$$\mathbb{W} = \frac{1}{N} \sum_{k=1}^{L} \left\{ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \hat{n}_{k+1} \check{n}_k \right\} \qquad \text{jump rates: } \frac{1}{N} = p = q$$

$$W(\vec{\hat{\rho}},\vec{\rho}) = \sum_{k=1}^{L} \Big[\rho_{k+1}(1-\rho_k)\underbrace{\left(e^{-(\hat{\rho}_{k+1}-\hat{\rho}_k)} - 1\right)}_{\simeq -(\hat{\rho}_{k+1}-\hat{\rho}_k)+\frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2} + \rho_k(1-\rho_{k+1})\underbrace{\left(e^{\hat{\rho}_{k+1}-\hat{\rho}_k} - 1\right)}_{\simeq (\hat{\rho}_{k+1}-\hat{\rho}_k)+\frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2} \Big]$$

This is the exact microscopic action.

We now perform a gradient expansion $(\hat{\rho}_{k+1}-\hat{\rho}_k \ll 1)$, yielding

$$W(\vec{\hat{\rho}},\vec{\rho}) = \sum_{k=1}^{L} -(\hat{\rho}_{k+1}-\hat{\rho}_k)\overbrace{\Big[\rho_{k+1}(1-\rho_k) - \rho_k(1-\rho_{k+1})\Big]}^{\rho_{k+1}-\rho_k} + \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2 \overbrace{\Big[\rho_{k+1}(1-\rho_k)+\rho_k(1-\rho_{k+1})\Big]}^{\simeq 2\rho_k(1-\rho_k)}$$

$$\boxed{\sigma(\rho) = 2\rho(1-\rho)}$$

This imposes to take the diffusive scaling limit:

$$\rho_{k+1}-\rho_k \mapsto \frac{1}{L}\partial_x \rho \qquad dt \mapsto L^2 dt \qquad \sum_{k=1}^{L} \to L\int_0^1 dx$$

$$\hat{\rho}_{k+1}-\hat{\rho}_k \mapsto \frac{1}{L}\partial_x \hat{\rho} \qquad \partial_t \mapsto L^{-2}\partial_t$$

$$S[\hat{\rho},\rho] = \int_0^t L^2 dt \left\{ \overbrace{\sum_{k=1}^{L}}^{\to L\int dx} \left[\hat{\rho}_k L^{-2}\partial_t \rho_k + \frac{1}{L}\partial_x\hat{\rho}\,\frac{1}{L}\partial_x\rho - \frac{1}{2}\sigma(\rho) L^{-2}(\partial_x\hat{\rho})^2 \right] \right\}$$

$$\boxed{S[\hat{\rho},\rho] = L\int_0^t dt \int_0^1 dx \left[\hat{\rho}\,\partial_t\rho + \partial_x\hat{\rho}\,\partial_x\rho - \frac{1}{2}\sigma(\rho)(\partial_x\rho)^2 \right]}$$

This action corresponds to a Langevin equation of the form

$$\partial_t \rho(x,t) = -\partial_x\left(-\partial_x\rho + \xi(x,t)\right), \qquad \langle \xi(x,t)\xi(x',t')\rangle = 2\,\frac{\sigma(\rho)}{L}\,\delta(x-x')\delta(t-t')$$

which describes the SSEP in the fluctuating hydrodynamic limit.

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Lecture on stochastic processes

- Another useful representation consists in setting a current $j(x,\tau)$ verifying the "continuity equation" (encoding conservation of particles)

$$\partial_t \rho + \partial_x j = 0 .$$

Then: $$S[\hat{\rho},\rho] = L \int_0^t d\tau \int_0^1 dx \left(\partial_x \hat{\rho} \left(j + \partial_x \rho \right) + \frac{1}{2} \sigma(\rho) (\nabla \hat{\rho})^2 \right)$$

Formally integrating over $\partial_x \hat{\rho}$ in $\int \mathcal{D}\hat{\rho}\, e^{-S[\rho,\hat{\rho}]}$ one find that

$$\boxed{\mathrm{Prob}\left[\rho(\tau)\right]_{\tau \le t} \approx \exp\left(-\frac{1}{2L} \int_0^t d\tau \int_0^1 dx \, \frac{\left(j + \partial_x \rho\right)^2}{2\sigma(\rho)} \right)}$$

which is another representation of the fluctuating hydrodynamics