Doi - Peliti Formalism for Lattice Gases -

- **Settings** : lattice with sites i

Each site is described by an occupation number $n_i = 0, 1, \dots$

Dynamics : on each site i, a particle jumps to any neighbor j with rate $w_{i\to j}(\vec{n})$ which depends on the configuration of the whole system $\vec{n} = \{n_i\}$

Hence, in terms of particle occupations, rates write

number of particles that can jump.

$$W(\dots n_i \dots n_j \dots \to \dots n_i - 1 \dots n_j + 1) = n_i \, w_{i\to j}(\vec{n})$$

Example : biased diffusion in dimension 1 $\begin{cases} w_{i\to i+1} = p & \text{jump to the right} \\ w_{i\to i-1} = q & \text{jump to the left} \end{cases}$

$$\begin{cases} W(\dots n_i n_{i+1} \dots) \to \dots n_i - 1 \; n_{i+1} + 1 \dots) = p \, n_i \\ W(\dots n_{i-1} n_i \dots) \to \dots n_{i-1} + 1 \; n_i - 1 \dots) = q \, n_i \end{cases}$$

- **Equation of evolution** : (boundary conditions may add other terms)

occupation number before the jump

$$\partial_t P(\vec{n}, t) = \sum_i \sum_{j \text{ neighb. } i} \Big\{ (n_i + 1) \, w_{i\to j}(\dots n_i + 1, n_j - 1 \dots) \, P(\dots n_i + 1, n_j - 1 \dots)$$
$$- n_i w_{i\to j}(\dots n_i \, n_j \dots) \, P(\dots n_i \, n_j \dots) \Big\} \quad (*)$$

Written in this form, it is quite complex to use.
One introduces a (Fock) base $|\vec{n}\rangle$ and $|P(t)\rangle = \sum_{\vec{n}} P(\vec{n}, t) |\vec{n}\rangle$
By changing variables in $(*)$ one has: $(n_i \mapsto n_i - 1 \, , \; n_j \mapsto n_j + 1)$

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \sum_i \sum_{j \text{ neighb. } i} \Big\{ n_i \, w_{i\to j}(\vec{n}) P(\vec{n}, t) \, |\dots n_i - 1, n_j + 1 \dots\rangle - n_i \, w_{i\to j}(\vec{n}) \, P(\vec{n}, t) \, |\vec{n}\rangle \Big\}$$

This constraint may be encoded in the $w_{i\to j}$'s.

Our aim is to encode this master equation in terms of creation & annihilation operators.

- **Doi-Peliti operators of creation and annihilation a and a^+**:

on a single site one defines : $\boxed{\begin{array}{l} a|n\rangle = n|n-1\rangle \quad (\text{with } a|0\rangle = 0) \\ a^+|n\rangle = |n+1\rangle \end{array}}$

on each site: $a_i|\vec{n}\rangle = n_i|\dots n_i - 1 \dots\rangle \qquad a_i^+|\vec{n}\rangle = |\dots n_i+1 \dots\rangle$

One also define a number operator $\hat{n}_i = a_i^+ a_i$, diagonal : $\hat{n}_i|\vec{n}\rangle = n_i|\vec{n}\rangle$

and also for a function $f(n)$ [$f(\vec{n})$] a diagonal operator $f(\hat{n})$ [$f(\hat{\vec{n}})$] such that $f(\hat{n}_i)|\vec{n}\rangle = f(n_i)|\vec{n}\rangle$ [$f(\hat{\vec{n}})|\vec{n}\rangle = f(\vec{n})|\vec{n}\rangle$]

- **Expression of the operator of evolution**:

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \sum_i \sum_{j \text{ neighb } i} \Big\{ \underbrace{\underbrace{w_{i\to j}(\vec{n})}\, P(\vec{n},t)\, a_i\, a_j^+ \underbrace{|\vec{n}\rangle}}_{= w_{i\to j}(\hat{\vec{n}})|\vec{n}\rangle} - w_{i\to j}(\vec{n})\, P(\vec{n},t) \quad |\vec{n}\rangle \Big\}$$

⚠ ordering ⚠ of $a, a^+, \hat{n}$

$$= \sum_i \sum_{j \text{ neighb } i} \left(a_j^+ a_i\, w_{i\to j}(\hat{\vec{n}}) - \hat{n}_i\, w_{i\to j}(\hat{\vec{n}}) \right) \underbrace{\sum_{\vec{n}} P(\vec{n},t)|\vec{n}\rangle}_{= |P(t)\rangle \text{ by definit}}$$

Finally : $$\boxed{\begin{array}{l} \partial_t |P(t)\rangle = \mathbb{W}|P(t)\rangle \text{ with} \\ \mathbb{W} = \displaystyle\sum_i \sum_{j \text{ neighb } i} \left\{ a_j^+ a_i\, w_{i\to j}(\hat{\vec{n}}) - \hat{n}_i\, w_{i\to j}(\hat{\vec{n}}) \right\} \end{array}}$$

This is the Doi-Peliti form of the operator of evolution

If one is able to diagonalize $\mathbb{W}$ one knows all the dynamics

<u>Remark</u>: The ordering is important.

<u>Generalisation</u>: If rates for particles are of the form $w_{i\to j}(\vec{n}) \cdot v_{i\to j}(\vec{n}')$ [$\vec{n}$: before the jump; $\vec{n}'$: after the jump]
i.e. depend on the system's state after the jump.

the term in the equat° for P is $(n_i+1)\, w_{i\to j}(n_i+1\ n_j-1)\, v_{i\to j}(n_i\, n_j)\, P(n_i+1\ n_j-1)$

— $|P\rangle$ is $n_i\, w_{i\to j}(\vec{n})\, v_{i\to j}(n_i-1\, n_{i+1}+1)\ |\dots n_i-1\ n_j+1 \dots\rangle$

— $\mathbb{W}$ is : $\boxed{\underbrace{v_{i\to j}(\hat{\vec{n}})}_{\text{after jump}}\ \underbrace{a_j^+ a_i}_{\text{jump}}\ \underbrace{w_{i\to j}(\hat{\vec{n}})}_{\text{before jump}}}$ → this allows to write the operator of evolution directly, without explicitating the master equation

- Including large deviation parameter s:

× 1^st case : s associated to $A = A_1$

One assumes that $A_1 \mapsto A_1 + a_1(i\to j)$ when a jump $i \to j$ occurs { which does not depend on the occupation numbers (this is easily generalized)

Then, in the equation for $\hat{P}(\vec{n}, s, t)$ one has:

$$\ldots (n_i+1)\, e^{-s a_1(i\to j)}\, w_{i\to j}(\ldots n_i+1, n_j-1 \ldots)\, P(\ldots n_i+1\; n_j-1 \ldots)$$

The operator of evolution is then:

{ If a_1 depends on the n_i's before the jump, this becomes $\hat{a}_1(i\to j; n)$.

$$\boxed{\mathbb{W}(s) = \sum_i \sum_{j\,\text{neighb}_i} \left\{ a_j^+ a_i\; e^{-s a_1(i\to j)}\, w_{i\to j}(\hat{\vec{n}}) - n_i w_{i\to j}(\hat{\vec{n}}) \right\}}$$

× 2^nd case : s associated to $A = A_2$

In this case, only the diagonal term of $\mathbb{W}$ is affected:

$$\boxed{\mathbb{W}(s) = \sum_i \sum_{j\,\text{neighb}_i} \left\{ a_j^+ a_i\, w_{i\to j}(\hat{\vec{n}}) - \left(\hat{n}_i w_{i\to j}(\hat{\vec{n}}) + s\, a_2(\hat{\vec{n}})\right) \right\}}$$

$\hat{n}_i = a_i^+ a_i$

- Remark on conservation of probability:

When $s=0$, one writes $\mathbb{W} = \sum_i \sum_{j n i} (a_j^+ - a_i^+)\, a_i\, w_{i\to j}(\hat{\vec{n}})$

We now use the identity $\langle n| a^+ = \langle n-1|$ (for a single site)) see later

$\langle \vec{n}| a_i^+ = \langle \ldots n_i - 1 \ldots |$ (for site i)) why.

Then $\underbrace{\langle - |}_{= \sum_n \langle n|} \mathbb{W} = \sum_{\vec{n}} \sum_{i\,\text{neighb}\,j} \langle - | (a_j^+ - a_i^+)\, a_i\, w_{ij}(\hat{\vec{n}}) = \sum_{\vec{n}} \sum_{ij} \left(\langle \ldots n_j - 1 \ldots | - \langle \ldots n_i - 1 \ldots | \right) a_i w_{ij}$

changing $\vec{n}$ as: $n_j \mapsto n_j + 1$, $n_i \mapsto n_i + 1$.

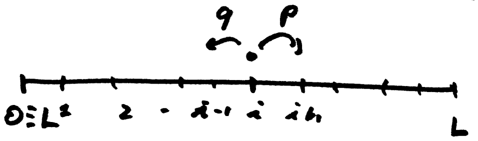
$$= \sum_{\vec{n}} \sum_{ij} \underbrace{\left(\langle n| - \langle n| \right)}_{0} a_i w_{ij}(\vec{n})$$

$\langle - | \mathbb{W} = 0$ Probability is conserved.

EXAMPLES - (At last!)

- (Biased) Diffusion in dimension 1 : with various boundary conditions

. periodic boundary conditions ,



a) $\mathbb{W} = \sum_{i=1}^{L} \left\{ q\, a_i^+ a_{i+1} + p\, a_{i+1}^+ a_i - q\, a_{i+1}^+ a_{i+1} - p\, a_i^+ a_i \right\}$

unbiased case: symmetric diffusion

$$\mathbb{W} = \sum_{i=1}^{L} a_i^+(a_{i+1} - a_i) + a_{i+1}^+(a_i - a_{i+1}) = \underbrace{\overbrace{-\sum_{i=1}^{L} (a_{i+1}^+ - a_i^+)(a_{i+1} - a_i)}^{\text{"}\nabla a^+ \nabla a\text{"}}}$$

$$\text{or: } \mathbb{W} = \sum_{i=1}^{L} a_i^+ (a_{i+1} + a_{i-1} - a_i - a_i) = \sum_{i=1}^{L} \underbrace{a_i^+ (a_{i+1} + a_{i-1} - 2a_i)}_{\text{"}a^+ \Delta a\text{"}}$$

b) With s conjugated to the current Q : $\quad Q \mapsto \begin{cases} Q+1 & \text{if } \curvearrowright \\ Q-1 & \text{if } \curvearrowleft \end{cases}$

$$\mathbb{W}(s) = \sum_{i=1}^{L} \left\{ e^{s} q\, a_i^+ a_{i+1} + e^{-s} p\, a_{i+1}^+ a_i - (p+q)\, a_i^+ a_i \right\}$$

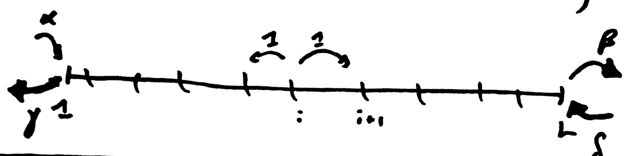
c) With s conjugated to the activity K : $\quad K \mapsto K+1$ at each jump

$$\mathbb{W}(s) = \sum_{i=1}^{L} \left\{ e^{s} \left[q\, a_i^+ a_{i+1} + p\, a_{i+1}^+ a_i \right] - (p+q)\, a_i^+ a_i \right\}$$

d) With s conjugated to the total particle number : $\hat{n}_{tot} = \sum_i \hat{n}_i \qquad N = \int_0^t d\tau \sum_i n_i$

$$\mathbb{W}(s) = \sum_{i=1}^{L} \left\{ q\, a_i^+ a_{i+1} + p\, a_{i+1}^+ a_i - (p+q)\, a_i^+ a_i - s\, a_i^+ a_i \right\}$$

- boundary condition: reservoirs



⚠ correct indicing is important!

$$\mathbb{W} = \underbrace{\sum_{i=1}^{L-1} \left\{ a_{i+1}^+ a_i + a_i^+ a_{i+1} - a_i^+ a_i - a_{i+1}^+ a_{i+1} \right\}}_{\substack{\text{bulk dynamics, written} \\ \text{link } i \leftrightarrow i+1 \text{ by link.}}} + \underbrace{\alpha(a_1^+ - 1) + \beta(a_L - a_L^+ a_L) + \gamma(a_1 - a_1^+ a_1) + \delta(a_L^+ - 1)}_{\text{contact with reservoirs}}$$

- Annihilation / creation :

On a single site : n = occupation number $n \in \mathbb{N}$

Dynamics : $\begin{cases} \cdot \text{ creation at rate } c \\ \cdot \text{ annihilation of each particle at rate } 1 \end{cases}$

Hence : for particle number $\begin{cases} W(n \to n+1) = c \\ W(n \to n-1) = n \end{cases}$

see the example at the end of part II. ⚠ where particle number was bounded

Equation for $P(n,t)$:

$$\partial_t P(n,t) = (n+1) P(n+1,t) + c\, P(n-1,t) - (n+c) P(n)$$

$$\partial_t |P\rangle = \sum_n \Big\{ n\, P(n,t) |n-1\rangle + \underbrace{c}_{r(\hat n)}\, P(n,t) |n+1\rangle - (n+c) P(n) |n\rangle \Big\}$$

$$\boxed{\mathbb{W} = a + c\, a^{\dagger} - (a^{\dagger} a + c)}$$

a and $a^{\dagger}$ directly represent reaction and annihilation

Remark : check that $\begin{cases} \cdot \text{ probability is conserved : } \langle - | \mathbb{W} = 0 \\ \cdot \text{ the equilibrium state is Poissonian: } |P_{eq}\rangle = e^{-c} \sum_{n \geq 0} \frac{c^n}{n!} |n\rangle \end{cases}$

- Reaction $A + A \xrightarrow{k} \emptyset$ (k is the reaction rate)

$$W(n \to n-2) = k\, n(n-1) \quad \begin{pmatrix} n: \text{ choice for the first particle} \\ n-1: \text{ 2nd} \end{pmatrix}$$

$$\partial_t P(n,t) = \sum_n \Big\{ k(n+2)(n+1) P(n+2,t) - k\, n(n-1) P(n,t) \Big\}$$

$$\partial_t |P\rangle = \sum_n \Big\{ k\, n(n-1) P(n) |n-2\rangle - k\, n(n-1) P(n) |n\rangle \Big\} \qquad [a, a^{\dagger}] = 1$$

$$\mathbb{W} = k\, a^2 - \underbrace{k\, a^{\dagger} a (a^{\dagger} a - 1)} \qquad \begin{cases} a a^{\dagger} = 1 + a^{\dagger} a \\ a^{\dagger} a (a^{\dagger} a - 1) = a^{\dagger} (1 + a^{\dagger} a) a - a^{\dagger} a = a^{\dagger 2} a^2 \end{cases}$$

This is $r(\hat n) = k\, \hat n (\hat n - 1)$

$$\boxed{\mathbb{W} = k (1 - a^{\dagger 2}) a^2}$$

Hence for instance the reaction-diffusion operator on a 1d lattice :

$$\mathbb{W} = - \sum_i (a_{i+1}^{\dagger} - a_i^{\dagger})(a_{i+1} - a_i) + k \sum_i (1 - a_i^{\dagger 2}) a_i^2$$

- Chemical reaction btw species : $\begin{cases} B \xrightarrow{k_1} A \\ A+B \xrightarrow{k_2} 2B \end{cases}$

operators a, a^+ for A
b, b^+ for B

think that this part is diagonal, and equal to the escape rate $k_1 \hat{n}_b$

$$\mathbb{W} = k_1 (a^+ b - b^+ b) + k_2 (b^{+2} ab - a^+ a\, b^+ b)$$

$$\boxed{\mathbb{W} = k_1 (a^+ - b^+) b + k_2 (b^{+2} - a^+ b^+) ab}$$ after some commutations

- Kinetically constrained models (KCM) (Bosonic version) on a 1d lattice

$\circ : n_i = 0$
$\bullet : n_i = 1$

$\circ \xrightarrow{c} \bullet$
$\bullet \xrightarrow{(1-c)} \circ$

with rates proportional to the number of active neighbors before the jump

$$\begin{cases} W(n_i \to n_i + 1) = c(n_{i+1} + n_{i-1}) \\ W(n_i \to n_i - 1) = (1-c)\, n_i\, (n_{i+1} + n_{i-1}) \end{cases}$$

$$\boxed{\mathbb{W} = \sum_i \left[c\, a_i^+ + (1-c) a_i - (c + (1-c) a_i^+ a_i) \right] \left(a_{i+1}^+ a_{i+1} + a_{i-1}^+ a_{i-1} \right)}$$

- Mean-Field version : n is the number of active sites

$$\begin{cases} W(n \to n+1) = c\, n \\ W(n \to n-1) = (1-c)\, n\, (n-1) \end{cases}$$

choice

$n-1$ neighbors to the one which decays

$$\mathbb{W} = c(a^{+2} a - a^+ a) + (1-c) \Big[\underbrace{a(a^+ a - 1)}_{a^+ a^2} + a^+ a \underbrace{(a^+ a - 1)}_{a^+ a^2} \Big]$$

$(a^+ a + 1) a - a$

$$\boxed{\mathbb{W} = c\, a^+ (a^+ - 1) a + (1-c)\, a^+ (1 + a^+) a^2}$$

• <u>Technical remark 1</u>: useful <u>similarity transformations</u>:

1) $\varphi(\ell) = e^{-\ell\hat{n}}\, a\, e^{\ell\hat{n}}$:
$$\begin{cases} \varphi(0) = a \\ \varphi'(\ell) = e^{-\ell\hat{n}} \overbrace{(-\hat{n}a + a\hat{n})}^{aa^\dagger a - a^\dagger a a = (aa^\dagger - a^\dagger a)a = a} e^{\ell\hat{n}} = e^{-\ell\hat{n}}\, a\, e^{\ell\hat{n}} = \varphi(\ell) \end{cases}$$
thus $\varphi(\ell) = e^{\ell}\varphi(0) = e^{\ell} a$ $\quad \boxed{e^{-\ell\hat{n}}\, a\, e^{\ell\hat{n}} = e^{\ell} a}$

2) $\varphi(\ell) = e^{-\ell\hat{n}}\, a^\dagger e^{\ell\hat{n}}$:
$$\begin{cases} \varphi(0) = a^\dagger \\ \varphi'(\ell) = e^{-\ell\hat{n}} \overbrace{(-\hat{n}a^\dagger + a^\dagger\hat{n})}^{a^\dagger a^\dagger a - a^\dagger a a^\dagger = a^\dagger(a^\dagger a - aa^\dagger) = -a^\dagger} e^{\ell\hat{n}} = -e^{-\ell\hat{n}} a^\dagger e^{\ell\hat{n}} = -\varphi(\ell) \end{cases}$$
thus $\varphi(\ell) = e^{-\ell}\varphi(0) = e^{-\ell} a^\dagger$ $\quad \boxed{e^{-\ell\hat{n}}\, a^\dagger e^{\ell\hat{n}} = e^{-\ell} a^\dagger}$

<u>1+2)</u> : One thus have found a similarity transformation $Q = e^{+\ell\hat{n}}$ such that
$$\boxed{\begin{cases} Q^{-1} a Q = z a \\ Q^{-1} a^\dagger Q = \frac{1}{z} a^\dagger \end{cases}}$$
(one has set $z = e^{\ell}$)

(Any similarity transformation does not modify the spectrum.)

3) $\varphi(\ell) = e^{-\ell a^\dagger}\, a\, e^{\ell a^\dagger}$
$$\left.\begin{cases} \varphi(0) = a \\ \varphi'(\ell) = e^{-\ell a^\dagger}(aa^\dagger - a^\dagger a)e^{\ell a^\dagger} = 1 \end{cases}\right\} \varphi(\ell) = a + \ell$$
in this case, setting $Q = e^{-\ell a^\dagger}$ one has
$$\boxed{\begin{cases} Q^{-1} a Q = a + \ell \\ Q^{-1} a^\dagger Q = a^\dagger \end{cases}}$$

4) Similarly $Q = e^{+\ell a}$ yields
$$\boxed{\begin{cases} Q^{-1} a Q = a \\ Q^{-1} a^\dagger Q = a^\dagger + \ell \end{cases}}$$

<u>Remark</u>: 1) and 2) can also be obtained directly:

$$z^{+\hat{n}}\, a\, z^{-\hat{n}} |n\rangle = z^{+\hat{n}} z^{-n} a|n\rangle = z^{+\hat{n}} z^{-n}\, n\, |n-1\rangle = z^{(n-1)} z^{-n} a|n\rangle = z^{-1} a |n\rangle$$
$$z^{-\hat{n}}\, a^\dagger z^{+\hat{n}} |n\rangle = z^{-\hat{n}} z^{n} |n+1\rangle = z^{-(n+1)} z^{n} |n+1\rangle = z^{-1} a^\dagger |n\rangle$$

↳ <u>Practical rule</u>: The spectrum of $\mathbb{W}(s)$ is not modified when performing $\begin{cases} a \mapsto z a \\ a^\dagger \mapsto z^{-1} a^\dagger \end{cases}$ and/or $a \mapsto a + \ell$ and/or $a^\dagger \mapsto a^\dagger + \ell$

(Those transformation are not unitary: in <u>Quantum Mechanics</u> they may change hermiticity)

• Technical remark 2: Consequence of particle conservation

Let's consider the (biased) walk 

with s conjugated $\begin{cases} \text{to the total current} \to \mathbb{W}_{tot}(s) \\ \text{to the current through the bond } L-1 \leftrightarrow L : \mathbb{W}_1(s) \end{cases}$

One has $\quad \mathbb{W}_{tot}(s) = \sum_{i=0}^{L-1} e^{s} q\, a_i^+ a_{i+1} + e^{-s} p\, a_{i+1}^+ a_i - q\, a_{i+1} a_{i+1} - p\, a_i^+ a_i$

$$\mathbb{W}_1(s) = \begin{cases} \sum_{i=0}^{L-2} q\, a_i^+ a_{i+1} + p\, a_{i+1}^+ a_i - q\, a_{i+1}^+ a_{i+1} - p\, a_i^+ a_i \\ + q e^{+s} a_{L-1}^+ a_L + p\, a_L^+ a_{L-1} e^{-s} - q\, \hat{n}_L - p\, \hat{n}_{L-1} \end{cases}$$

↪ NB $a_L^+ = a_0^+ \quad a_L = a_0$

One sets $\quad Q = e^{s \sum_{i=0}^{L-1} i \hat{n}_i}$ (Reminiscent of a "Jordan-Wigner" transform in quantum mech.)

one has for $0 \leq i \leq L-2$ $\begin{cases} Q^{-1} a_i^+ a_{i+1} Q = e^{is} e^{-(i+1)s} a_i^+ a_{i+1} = e^{-s} a_i^+ a_{i+1} \\ Q^{-1} a_{i+1}^+ a_i Q = e^{(i+1)s} e^{-is} a_{i+1}^+ a_i = e^{s} a_{i+1}^+ a_i \end{cases}$

But: for $i = L-1$: $\quad Q^{-1} a_{L-1}^+ a_L Q = Q^{-1} a_{L-1}^+ a_0 Q = e^{(L-1)s} a_{L-1}^+ a_0$

$$Q^{-1} a_L^+ a_{L-1} Q = Q^{-1} a_0^+ a_{L-1} Q = e^{-(L-1)s} a_0^+ a_{L-1}$$

Hence: $\quad Q^{-1} \mathbb{W}_{tot}(s) Q = \sum_{i=0}^{L-2} q\, a_i^+ a_{i+1} + p\, a_{i+1}^+ a_i - q\, a_{i+1}^+ a_{i+1} - p\, a_i^+ a_i$

$$+ q e^{Ls} a_{L-1}^+ a_L + p e^{-Ls} a_L^+ a_{L-1} - q\, \hat{n}_L - p\, \hat{n}_{L-1}$$

$Q^{-1} \mathbb{W}_{tot}(s) Q = \mathbb{W}_1(Ls)$ $\quad$ thus, since spectra are the same:

$$\boxed{\psi_{tot}(s) = \psi_1(Ls)}$$

We thus have shown that the large deviation function for the total current is simply related to that of the current through one bond. <u>This result is not trivial</u>

• Technical remark 3:

a^{+} is not the adjoint of a, contrary to the situation in quantum mech.

× Reminder on quantum mechanics:

The quantum creation and annihilation operators (denoted α, α^{+} here) are such that $\boxed{\begin{cases} \alpha|n\rangle = \sqrt{n}\,|n-1\rangle \\ \alpha^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}}$

α^{+} is the hermitian conjugate of α. Indeed: using $\mathbb{1} = \sum_m |m\rangle\langle m|$

$$\langle n|\alpha^{+} = \langle n|\underbrace{\alpha^{+}\sum_m |m\rangle}\langle m| = \sum_m \underbrace{\langle n|\sqrt{m+1}\,|m+1\rangle}_{\delta_{m+1,n}}\langle m| = \sqrt{n}\,\langle n-1| = \left(\sqrt{n}\,|n-1\rangle\right)^{+}$$

$$\boxed{\langle n|\alpha^{+} = \left(\alpha|n\rangle\right)^{+}}$$

× Similarity transformation from α, α^{+} to a, a^{+}:

The quantum operator number $\alpha^{+}\alpha$ is equal to $a^{+}a = \hat{n}$: $\alpha^{+}\alpha|n\rangle = \alpha^{+}\sqrt{n}\,|n-1\rangle = n|n\rangle$ $\Rightarrow$ $\alpha^{+}\alpha|n\rangle = \hat{n}|n\rangle$

One sets: $\boxed{\begin{cases} a = (\hat{n}!)^{-1/2}\,\alpha\,(\hat{n}!)^{1/2} \\ a^{+} = (\hat{n}!)^{-1/2}\,\alpha^{+}(\hat{n}!)^{1/2} \end{cases}}$

Here $\hat{n}! = n!$ on $|n\rangle$. and $\hat{n}^{+} = \hat{n}$

Then: $$a|n\rangle = (\hat{n}!)^{-1/2}(n!)^{1/2}\sqrt{n}\,|n-1\rangle = \underbrace{\left(\frac{n!}{(n-1)!}\right)^{1/2}}_{\sqrt{n}}\sqrt{n}\,|n-1\rangle = n|n-1\rangle \text{ ok}$$

$$a^{+}|n\rangle = (\hat{n}!)^{-1/2}(n!)^{1/2}\sqrt{n+1}\,|n+1\rangle = \underbrace{\left(\frac{n!}{(n+1)!}\right)^{1/2}}_{1/\sqrt{n+1}}\sqrt{n+1}\,|n+1\rangle = |n+1\rangle \text{ ok}$$

Rq: one also has $\begin{cases} \alpha = (\hat{n}!)^{1/2}\,a\,(\hat{n}!)^{-1/2} \\ \alpha^{+} = (\hat{n}!)^{1/2}\,a^{+}(\hat{n}!)^{-1/2} \end{cases}$

The expressions above are thus correct representations of a, a^{+}

→ In other words the (statistical physics) Doi-Peliti operators a, a^{+} are (non-unitary) similarity transformed of the quantum creation and annihilation operators α, α^{+}.

The non-unitarity explains why $(a)^{+} \neq a^{+}$.

Indeed: $(a)^{+} = \left((\hat{n}!)^{-1/2}\alpha(\hat{n}!)^{1/2}\right)^{+} = (\hat{n}!)^{1/2}\underbrace{(\alpha)^{+}}_{=\alpha^{+}}(\hat{n}!)^{-1/2} = (\hat{n}!)^{1/2}(\hat{n}!)^{1/2}a^{+}(\hat{n}!)^{-1/2}(\hat{n}!)^{-1/2}$ $\Rightarrow$ $\boxed{(a)^{+} = \hat{n}!\,a^{+}(\hat{n}!)^{-1}}$

Lecture on Stoch. process

- Technical remark 4: how can one transpose a, a^+ ?

$$\langle n|a^+ = \langle n|a^+ \overbrace{\sum_m |m\rangle\langle m|}^{\mathbb{1}\text{ (identity)}} = \sum_m \overbrace{\langle n|m+1\rangle}^{\delta_{n,m+1}}\langle m| = \langle n-1| \Rightarrow \boxed{\langle n|a^+ = \langle n-1|}$$

$$\langle n|a = \langle n|a\sum_m |m\rangle\langle m| = \sum_m \overbrace{\langle n|m|m-1\rangle}^{m\,\delta_{n,m-1}}\langle m| = \langle n+1|(n+1) \Rightarrow \boxed{\langle n|a = \langle n+1|(n+1)}$$

- Technical remark 5: What is the meaning of the eigen vectors associated to $\mathbb{W}(s)$?

. Let's call $|R\rangle$ and $\langle L|$ the left and right eigen vectors associated to the maximum eigen value $\psi(s)$ of $\mathbb{W}(s)$

One knows that at $s=0$: $\langle L| = \langle -| = \sum_{\mathcal{C}}\langle \mathcal{C}|$ and $|R\rangle = |P_{st}\rangle$ (steady state).

. One can normalize $|R\rangle$ such that $\langle -|R\rangle = 1$ (ie $\sum_{\mathcal{C}} R(\mathcal{C}) = 1$: R is normalized).

Consider an observable $O(\mathcal{C})$ depending on the configuration.

. Final-time average in the s-state:

diagonal operator of elements $O(\mathcal{C})$: $\hat{O} = \sum_{\mathcal{C}} O(\mathcal{C})|\mathcal{C}\rangle\langle\mathcal{C}|$

$$\langle O\rangle_s^{final} = \lim_{t\to\infty}\langle O(\mathcal{C}(t))\rangle_s = \lim_{t\to\infty}\frac{\langle -|\hat{O}\,e^{t\mathbb{W}(s)}|P_0\rangle}{\langle -|\underbrace{e^{t\mathbb{W}(s)}|P_0\rangle}_{\text{in the long time limit: } \sim e^{t\psi(s)}|R\rangle}}$$

P_0 ← initial configuration

$$= \frac{\langle -|\hat{O}\,e^{t\psi(s)}|R\rangle}{\langle -|\,e^{t\psi(s)}|R\rangle} \quad \text{Hence: } \boxed{\langle O\rangle_s^{final} = \langle -|\hat{O}|R\rangle}$$

. Intermediate-time average in the s-state: let "$0 \ll \tau \ll t$"

$$\langle O\rangle_s^{interm.} = \lim_{\tau\to\infty}\lim_{t\to\infty}\langle O(\mathcal{C}(\tau))\rangle_s = \lim_{\tau\to\infty}\lim_{t\to\infty}\frac{\langle -|e^{(t-\tau)\mathbb{W}(s)}\hat{O}\,e^{\tau\mathbb{W}(s)}|P_0\rangle}{\underbrace{\langle -|e^{(t-\tau)\mathbb{W}(s)}}_{\sim\langle L|e^{(t-\tau)\psi(s)}}\;\underbrace{e^{\tau\mathbb{W}(s)}|P_0\rangle}_{\sim e^{\tau\psi(s)}|R\rangle}}$$

$$\boxed{\langle O\rangle_s^{final} = \langle L|\hat{O}|R\rangle}$$

. Other demonstration: consider the average of the time-integrated O:

$$\langle O\rangle_s^{integ} = \lim_{t\to\infty}\left\langle \frac{1}{t}\int_0^t d\tau\, O(\mathcal{C}(\tau))\right\rangle_s = \lim_{t\to\infty}\partial_t\partial_h\Big|_{h=0}\langle -|\underbrace{e^{t(\mathbb{W}(s)+h\hat{O})}}|P_0\rangle$$

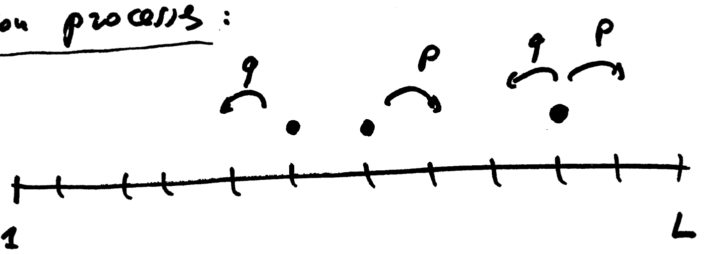
$$= \langle L|\hat{O}|R\rangle$$

One uses here a perturbation theory at first order in h

Lecture 3, part II : Operator Formalism for Exclusion Processes

• **MOTIVATIONS :**

Exclusion processes :



× Each site i is either empty ($n_i = 0$) or occupied ($\bullet$, $n_i = 1$) by a particle
Each particle can jump to its left (right) with rate q (p)
provided the target site is empty (EXCLUSION RULE)

× This can model :
- car / pedestrian (in a queue) traffic, jam
- motion of motor proteins (e.g.: dynein, kinesin) on microtubules

× Classification :

SSEP	(symmetric)	$p = q$
ASEP	(asymmetric)	$p \neq q$
TASEP	(totally –)	$q = 0$ $p > 0$
WASEP	(weakly asymmetric)	$p - q = \frac{E}{L}$

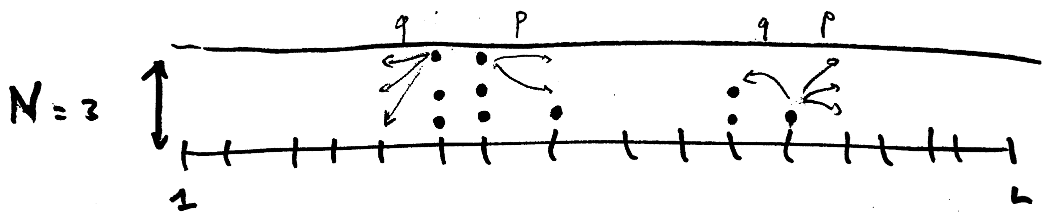
× This is the "Ising model" of non-equilibrium dynamics
The (non-equilibrium) steady state (NESS) is known in varying situations
- closed boundary condition, or periodic
- contact with reservoirs at the boundaries

The large deviation of current / activity have been computed in the large size limit (with finite size correction) in the (WA)SEP.

- Partial Exclusion Process :

The exclusion condition is relaxed : each site can contain between 0 and $N < \infty$ particles

Each particle can jump to any of the empty site with rate p or q



in other words, in terms of particle number transition rates:

$$\boxed{\begin{aligned} W(\dots n_i\, n_{i+1} \dots \xrightarrow{p\ \text{(jump to the right)}} \dots n_i - 1\ n_{i+1}+1 \dots) &= p\, \underset{\uparrow}{n_i}\, (N - n_{i+1}) \\ W(\dots n_i\, n_{i+1} \dots \xrightarrow{q} \dots n_i + 1\ n_{i+1} - 1 \dots) &= q\, (N - n_i)\, n_{i+1} \end{aligned}}$$

choice of the moving particle; $(N-n_{i+1})$ ← choice of the target site; n_{i+1} ← particle moving; $(N-n_i)$ ↑ target site

This implements the partial exclusion ($N - n = 0$ if the occupation # n of the target site is $n = N$ i.e. if the target site is full)

- Spin operators for stochastic processes :

One defines, for each site described by a vector $|n\rangle$ $(0 \leq n \leq N)$

$$\boxed{\begin{aligned} S^+ |n\rangle &= (N-n)\, |n+1\rangle \\ S^- |n\rangle &= n\, |n-1\rangle \\ \hat{n}\, |n\rangle &= n\, |n\rangle \end{aligned}}$$

or matricially: $\hat{n} = \begin{pmatrix} 0 & & (0) \\ & \ddots & \\ (0) & & N \end{pmatrix}$

$S^+ = \begin{pmatrix} 0 & & & (0) \\ N & \ddots & & \\ & \ddots & \ddots & \\ (0) & & 1 & 0 \end{pmatrix}$ $\quad S^- = \begin{pmatrix} 0 & 1 & & (0) \\ & \ddots & \ddots & \\ & & \ddots & N \\ (0) & & & 0 \end{pmatrix}$

One also reads from the matrices :

$$\left.\begin{aligned} \langle n | S^+ &= \langle n-1 | (N - n + 1) \\ \langle n | S^- &= \langle n+1 | (n+1) \end{aligned}\right\}$$

where by convention one notes :

$$\langle -1 | = \langle N+1 | = 0$$

3.13

Lectures on Stoch processes
2012

- Commutation relations:

One defines $S^z = \hat{n} - \frac{1}{2}N = \begin{pmatrix} -\frac{N}{2} & & (0) \\ & \ddots & \\ (0) & & \frac{N}{2} \end{pmatrix}$

One checks directly that $S^\pm, S^z$ obey the commutation relations

$$\boxed{[S^z, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = 2S^z}$$

which are the same for quantum operators

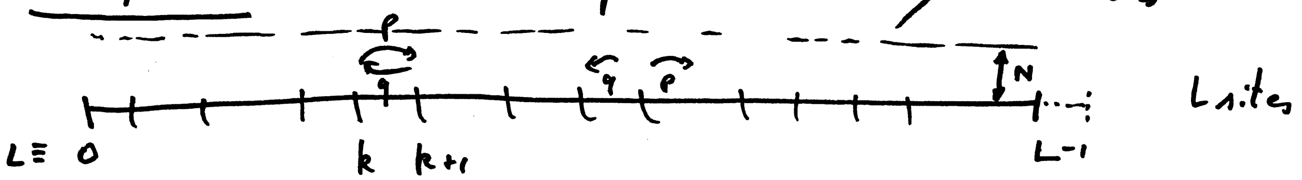
Similarly, defining $S^x = \frac{1}{2}(S^+ + S^-)$, $S^y = \frac{1}{2i}(S^+ - S^-)$ one has $\boxed{[S^x, S^y] = i S^z}$ and similar cyclic permutation (i: complex number $i^2 = -1$)

Let's e.g show that $[S^+, S^-] = 2S^z = 2\hat{n} - N$:

$$[S^+, S^-]|n\rangle = S^+ n|n-1\rangle - S^-(N-n)|n+1\rangle = [\underbrace{(N-n+1)n - (N-n)(n+1)}_{Nn - n^2 + n - Nn - N + n^2 + n}]|n\rangle$$

$= (2n - N)|n\rangle$ hence the result

- Example 1: ASEP with periodic boundary conditions



$$\partial_t P(\vec{n}, t) = \sum_{k=0}^{L-1} \Big\{ p(n_k+1)(N-n_{k+1}-1) P(n_k+1, n_{k+1}-1, t) + q(N-n_k+1)(n_{k+1}+1) P(n_k-1, n_{k+1}+1, t)$$
$$- q\, n_{k+1}(N-n_k) P(\vec{n}) - p\, n_k (N-n_{k+1}) P(\vec{n}) \Big\}$$

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \sum_{k=0}^{L-1} \Big\{ \overbrace{p(n_k+1)(N-n_{k+1}-1) P(n_k+1, n_{k+1}-1, t)}^{n_k \mapsto n_k - 1 \,;\, n_{k+1} \mapsto n_{k+1}+1} |\vec{n}\rangle + \overbrace{q(N-n_k+1)(n_{k+1}+1) P(n_k-1, n_{k+1}-1, t)}^{n_k \mapsto n_k + 1 \quad n_{k+1} \mapsto n_{k+1} - 1} |\vec{n}\rangle$$
$$- q\, n_{k+1}(N-n_k) P(\vec{n}) |\vec{n}\rangle - p\, n_k (N - n_{k+1}) P(\vec{n}) |\vec{n}\rangle \Big\}$$

change of variable on $\vec{n}$ so as to factorize by $P(\vec{n})$

$$= \sum_{\vec{n}} \sum_{k=0}^{L-1} \Big\{ p\, n_k (N-n_{k+1}) |n_k - 1, n_{k+1}+1\rangle + q(N-n_k) n_{k+1} |n_k+1, n_{k+1}-1\rangle$$
$$- p\, n_k (N - n_{k+1}) |\vec{n}\rangle - q(N-n_k) n_{k+1} |\vec{n}\rangle \Big\} P(\vec{n})$$

in the sums: $L \equiv 0$ (periodic b.c.)

In the gain term of that equation, one reckognizes

$$\begin{cases} n_k (N-n_{k+1}) \,|n_k - 1, n_{k+1}+1\rangle = S_k^- S_{k+1}^+ \,|\vec{n}\rangle \\ (N-n_k) n_{k+1} \,|n_k + 1, n_{k+1} - 1\rangle = S_k^+ S_{k+1}^- \,|\vec{n}\rangle \end{cases}$$

one notes $\hat{n}_k : \hat{n}_k |\vec{n}\rangle = n_k$

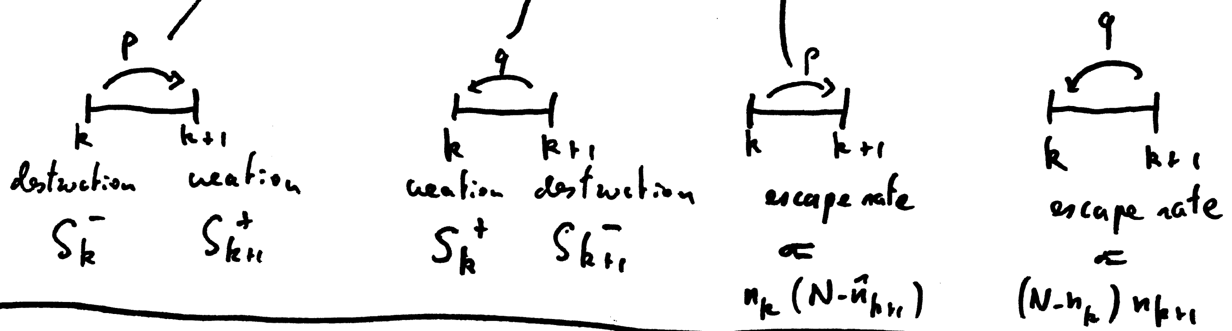
$$\boxed{\check{n}_k = N - \hat{n}_k}$$

<u>Thus:</u>

$$\partial_t |P(t)\rangle = \sum_{k=0}^{L-1} \left\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\} \sum_{\vec{n}} P(\vec{n}) |\vec{n}\rangle$$

One identifies $\mathbb{W}$ in $\partial_t |P(t)\rangle = \mathbb{W} |P(t)\rangle$ to:

$$\boxed{\mathbb{W} = \sum_{k=0}^{L-1} \left\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\}}$$



One thus reads directly from the dynamics the form of $\mathbb{W}$

- <u>Example 1^{bis}</u>: s-modified operator of evolution $\mathbb{W}(s)$

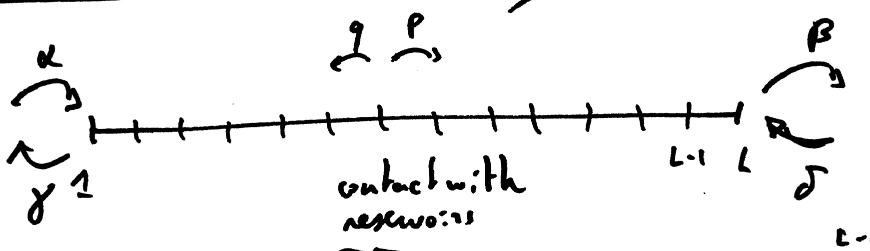
  - <u>s conjugated to the activity K</u>:

$$\mathbb{W}(s) = \sum_{k=0}^{L-1} \left\{ e^{-s} \left[p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- \right] - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\}$$

  - <u>s conjugated to the total current Q</u>:

$$\mathbb{W}(s) = \sum_{k=0}^{L-1} \left\{ e^{-s} p\, S_k^- S_{k+1}^+ + e^{s} q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \right\}$$

- Example 2: with boundary conditions with reservoirs



$$\mathbb{W} = \mathbb{W}_{bulk} + \widetilde{\mathbb{W}}_{res} \quad \text{where} \quad \mathbb{W}_{bulk} = \sum_{k=1}^{L-1} \left\{ p S_k^- S_{k+1}^+ + q S_k^+ S_{k+1}^- - p \hat{n}_k \check{n}_{k+1} - q \check{n}_k \hat{n}_{k+1} \right\}$$

× from the rules given previously one expects:

$$\mathbb{W}_{res} = \alpha (S_1^+ - \check{n}_1) + \delta (S_L^+ - \check{n}_L) + \gamma (S_1^- - \hat{n}_1) + \beta (S_L^- - \hat{n}_L)$$

× let's for instance check explicitly the left $\binom{\alpha}{\gamma}$ boundary term:

$$\partial_t P(\vec{n},t) \overset{\alpha\gamma}{=} \alpha(N-n_1+1) P(n_1-1,t) + \gamma(n_1+1) P(n_1+1,t) - \alpha(N-n_1) P(n_1,t) - \gamma n_1 P(n_1,t)$$

$$\partial_t |P(t)\rangle = \sum_{\vec{n}} \Big(\overbrace{\alpha(N-n_1+1) P(n_1-1,t)|\vec{n}\rangle}^{n_1 \mapsto n_1+1} + \overbrace{\gamma(n_1+1) P(n_1+1,t)|\vec{n}\rangle}^{n_1 \mapsto n_1-1} - \alpha(N-n_1) P(n_1,t)|\vec{n}\rangle - \gamma n_1 P(n_1,t)|\vec{n}\rangle \Big)$$

change of variable on $\vec{n}$ so as to factorize $P(\vec{n},t)$

$$= \sum_{\vec{n}} \Big(\overbrace{\alpha(N-n_1)|n_1+1\rangle}^{S_1^+|\vec{n}\rangle} + \overbrace{\gamma n_1 |n_1-1\rangle}^{S_1^-|\vec{n}\rangle} - \alpha(N-n_1)|\vec{n}\rangle - \gamma n_1 |\vec{n}\rangle \Big) P(\vec{n},t)$$

$$= \underbrace{\Big(\alpha(S_1^+ - \check{n}_1) + \gamma(S_1^- - \hat{n}_1) \Big)}_{} \underbrace{\sum_{\vec{n}} P(\vec{n},t)|\vec{n}\rangle}_{|P(t)\rangle}$$

this is the left term of the boundary operator $\mathbb{W}_{res}$ that we expected

- Analogy with quantum mechanics

What is the link between the $S^{\pm}$, S^z and the quantum ones?

- Quantum operators: $\Sigma^{\pm}$, Σ^z verify

$$\begin{cases} \Sigma^+ |n\rangle = \sqrt{(n+1)(N-n)}\ |n+1\rangle \\ \Sigma^- |n\rangle = \sqrt{n(N-n+1)}\ |n-1\rangle \\ \Sigma^z |n\rangle = (n - \frac{1}{2}N)\ |n\rangle \end{cases} \quad \rightarrow \Sigma^z = S^z$$

Besides, they are hermitian-adjoint: $(\Sigma^+)^\dagger = \Sigma^-$

- From quantum to statistical operators:

Consider $\boxed{Q = \binom{N}{\hat{n}}^{-1/2}}$ (binomial coefficient) $\binom{N}{n} = \dfrac{N!}{n!\,(N-n)!}$

and let's show that

$$\boxed{S^{\pm} = Q^{-1}\Sigma^{\pm} Q}$$

$$\frac{(n+1)!\,(N-n-1)!}{n!\,(N-n)!} = \frac{n+1}{N-n}$$

- $Q^{-1}\Sigma^+ Q\,|n\rangle = \binom{N}{n}^{-1/2} Q^{-1}\sqrt{(n+1)(N-n)}\ |n+1\rangle = \sqrt{(n+1)(N-n)}\left(\dfrac{\binom{N}{n}}{\binom{N}{n+1}}\right)^{-1/2} |n+1\rangle$

$$= \sqrt{\frac{(n+1)(n+1)^{-1}(N-n)}{(N-n)^{-1}}}\ |n+1\rangle = (N-n)\,|n+1\rangle = S^+|n\rangle \quad \underline{\text{OK}}$$

- $Q^{-1}\Sigma^- Q\,|n\rangle = \binom{N}{n}^{-1/2} Q^{-1}\sqrt{n(N-n+1)}\ |n-1\rangle = \sqrt{n(N-n+1)}\left(\dfrac{\binom{N}{n}}{\binom{N}{n-1}}\right)^{-1/2} |n-1\rangle$

$$= \sqrt{\frac{n(N-n+1)\,n}{N-n+1}}\ |n-1\rangle \qquad \left(\frac{n!\,(N-n)!}{(n-1)!\,(N-n+1)!}\right)^{-1} = \left(\frac{n}{N-n+1}\right)^{-1}$$

$$= n\,|n-1\rangle = S^-|n-1\rangle \quad \underline{\text{OK}}$$

- To summarize: $S^{\pm}$ are (non-unitary) similarity transformations through Q of the quantum spin operators $\Sigma^{\pm}$

Since the transformation is not unitary: $(S^+)^\dagger \neq S^-$ while $(\Sigma^+)^\dagger = \Sigma^-$

Indeed $(S^+)^\dagger = Q^\dagger \Sigma^- Q^{-\dagger} = \underbrace{Q^\dagger Q}_{\neq \mathbb{1}} S^- \underbrace{Q^{-1} Q^{-\dagger}}_{\neq \mathbb{1}}$ since Q is not unitary

• **Steady state** in equilibrium for the symmetric S.E.P.

Fix a density $0<\rho<1$ and consider the eq. distrib. product of Bernoulli laws

$$P^\rho_{eq}(\vec n) = \prod_{k=1}^{L} \binom{N}{n_k} \rho^{n_k}(1-\rho)^{N-n_k}$$

× case of a single site : $|B^\rho\rangle = \sum_{n=0}^{N} P^\rho_{eq}(n)|n\rangle = \sum_{n=0}\binom{N}{n}\rho^n(1-\rho)^{N-n}|n\rangle$

action of S^+ :

$$(N-n+1)\frac{N!}{(n-1)!(N-n+1)!} = n\frac{N!}{n!(N-n)}$$

$$S^+|B^\rho\rangle = \sum_{n=0}^{N}\binom{N}{n}\rho^n(1-\rho)^{N-n}(N-n)|n+1\rangle \overset{n\to n-1}{=} \sum_{n=0}\binom{N}{n-1}\rho^{n-1}(1-\rho)^{N-n+1}(N-n+1)|n\rangle$$

$$= \sum_{n=0}^{N} n\,\frac{1-\rho}{\rho}\binom{N}{n}\rho^n(1-\rho)^{N-n}|n\rangle \qquad \text{we used } (N-n+1)\binom{N}{n-1} = n\binom{N}{n}$$

$$= \frac{1-\rho}{\rho}\sum_{n=0}^{N}\binom{N}{n}\rho^n(1-\rho)^{N-n}\underbrace{n|n\rangle}_{\hat n|n\rangle} = \frac{1-\rho}{\rho}\,\hat n\,|B^\rho\rangle$$

$$\boxed{S^+|B^\rho\rangle = \frac{1-\rho}{\rho}\,\hat n\,|B^\rho\rangle}$$

$$(n+1)\frac{N!}{(n+1)!(N-n-1)!} = (N-n)\frac{N!}{n!(N-n)!}$$

action of S^- :

$$S^-|B^\rho\rangle = \sum_{n=0}^{N}\binom{N}{n}\rho^n(1-\rho)^{N-n}\,n|n-1\rangle = \sum_{n=0}^{N}\binom{N}{n+1}\rho^{n+1}(1-\rho)^{N-n-1}(n+1)|n\rangle$$

$$= \sum_{n=0}^{N}(N-n)\left(\frac{1-\rho}{\rho}\right)^{-1}\binom{N}{n}\rho^n(1-\rho)^{N-n}|n\rangle \qquad \text{where we used } (n+1)\binom{N}{n+1} = (N-n)\binom{N}{n}$$

$$= \frac{\rho}{1-\rho}\sum_{n=0}^{N}\binom{N}{n}\rho^n(1-\rho)^{N-n}(N-n)|n\rangle = \frac{\rho}{1-\rho}(N-\hat n)|B^\rho\rangle$$

$$\boxed{S^-|B^\rho\rangle = \frac{\rho}{1-\rho}\,\check n\,|B^\rho\rangle}$$

× case of $1 \cdots L$ with p.b.c and $p=q$: $\mathbb{W} = \sum_{k=1}^{L}\{S_k^+S_{k+1}^- + S_k^-S_{k+1}^+ - \hat n_k\check n_{k+1} - \check n_k\hat n_{k+1}\}$

Product law on the sites

Thus $$\mathbb{W}|B^\rho\rangle = \sum_{k=1}^{L}\Big\{\underbrace{\frac{\rho}{1-\rho}\frac{1-\rho}{\rho}}_{=1}\hat n_k\check n_{k+1} + \underbrace{\frac{\rho}{1-\rho}\frac{1-\rho}{\rho}}_{=1}\hat n_{k+1}\check n_k - \hat n_k\check n_{k+1} - \check n_k\hat n_{k+1}\Big\}|B^\rho\rangle$$

and one finds $\mathbb{W}|B^\rho\rangle = 0$: $\boxed{\text{$|B^\rho\rangle$ is the steady state with ρ fixed by the total \# of particles}}$

One should constrain total # of particle →

- Eq. steady state in contact with reservoirs:

Now, the # of particle is not fixed.
Let's see how $\alpha\, \beta\, \gamma\, \delta$ are determined, and fix ρ:

$$\mathbb{W}_{\text{reservoirs}} = \alpha\left(S_1^+ - \check{n}_1\right) + \delta\left(S_L^+ - \check{n}_L\right) + \gamma\left(S_1^- - \hat{n}_1\right) + \beta\left(S_L^- - \hat{n}_L\right)$$

$$\mathbb{W}_{res}|B\rho\rangle = \alpha\left(\frac{1-\rho}{\rho}\hat{n}_1 - \check{n}_1\right) + \delta\left(\frac{1-\rho}{\rho}\hat{n}_L - \check{n}_L\right) + \gamma\left(\frac{\rho}{1-\rho}\check{n}_1 - \hat{n}_1\right) + \beta\left(\frac{\rho}{1-\rho}\check{n}_L - \hat{n}_L\right)$$

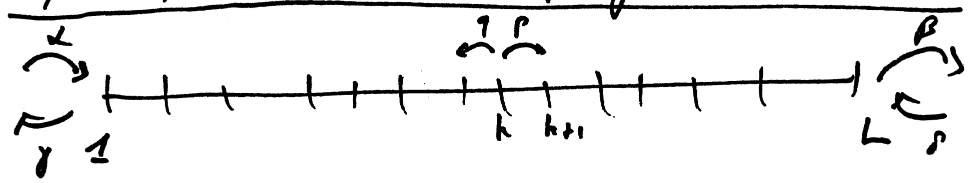
$$= \left(\alpha\frac{1-\rho}{\rho} - \gamma\right)\hat{n}_1 + \left(\delta\frac{1-\rho}{\rho} - \beta\right)\hat{n}_L + \left(\gamma\frac{\rho}{1-\rho} - \alpha\right)\underbrace{\check{n}_1}_{(N-\hat{n}_1)} + \left(\beta\frac{\rho}{1-\rho} - \delta\right)\underbrace{\check{n}_L}_{(N-\hat{n}_L)}$$

For all those terms to vanish one needs:

$$\begin{cases} \alpha\dfrac{1-\rho}{\rho} = \gamma \\ \delta\dfrac{1-\rho}{\rho} = \beta \end{cases} \quad \text{ie} \quad \boxed{\begin{array}{c} \rho = \dfrac{\alpha}{\alpha+\gamma} \\ \hline \rho = \dfrac{\delta}{\delta+\beta} \end{array}} \text{ equality implies } \frac{\alpha}{\gamma} = \frac{\delta}{\beta}$$

> The solution for the steady state is thus $|B\rho\rangle$ with $\rho = \dfrac{\alpha}{\alpha+\gamma} = \dfrac{\delta}{\delta+\beta}$ provided $\dfrac{\alpha}{\gamma} = \dfrac{\delta}{\beta}$

- Symmetry 1 : conservation of particle number :



Let's relate the operator $\mathbb{W}_{tot}(s)$ for the total current to the one $\mathbb{W}_1(s)$ for the current at the right boundary :

$$\mathbb{W}_{tot}(s) = \sum_{k=1}^{L-1} \{ p\, S_k^- \underbrace{S_{k+1}^+ e^{-s}}_{\text{jump} \leftarrow} + q\, S_k^+ \underbrace{S_{k+1}^- e^{s}}_{\text{jump} \rightarrow} - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \} + \alpha(e^{s} S_1^+ - \check{n}_1) + \delta(S_L^+ e^{s} - \check{n}_L) + \gamma(e^{s} S_1^- - \hat{n}_1) + \beta(S_L^- e^{-s} - \hat{n}_L)$$

× Let's find a similarity transformat° Q such that $Q^{-1} S^\pm Q = (z)^{\pm 1} S^\pm$:

$$z^{+\hat{n}} S^+ z^{-\hat{n}} |n\rangle = z^{n} z^{(n+1)} (N-n) |n+1\rangle = z\, S^+ |n\rangle$$
$$z^{-\hat{n}} S^- z^{\hat{n}} |n\rangle = z^{n} z^{-(n+1)}\, n\, |n-1\rangle = z^{-1} S^- |n\rangle$$

thus $Q = z^{-\hat{n}}$ works

× Apply $Q^{-1} \cdot Q$ with $Q = e^{-s \sum_{k=1}^{L} k \hat{n}_k}$:

using

$$Q^{-1} S_k^- S_{k+1}^+ Q = e^{-sk} e^{s(k+1)} S_k^- S_{k+1}^+ = e^{s} S_k^- S_{k+1}^+$$
$$Q^{-1} S_k^+ S_{k+1}^- Q = e^{sk} e^{-s(k+1)} S_k^+ S_{k+1}^- = e^{-s} S_k^+ S_{k+1}^-$$

$$Q^{-1} S_1^\pm Q = e^{\pm s} S_1^\pm$$
$$Q^{-1} S_L^\pm Q = e^{\pm L s} S_L^\pm$$

One finds :

$$Q^{-1} \mathbb{W}_{tot}(s) Q = \sum_{k=1}^{L-1} \overbrace{\{ p\, S_k^- S_{k+1}^+ + q\, S_k^+ S_{k+1}^- - p\, \hat{n}_k \check{n}_{k+1} - q\, \check{n}_k \hat{n}_{k+1} \}}^{s \text{ has disappeared from the bulk}} + \alpha(S_1^+ - \check{n}_1) + \underbrace{\delta(e^{s(L+1)} S_L^+ - \check{n}_L)}_{\text{and remains at the right boundary}} + \gamma(S_1^- - \hat{n}_1) + \beta(e^{-s(L+1)} S_L^- - \hat{n}_L)$$

$$\boxed{Q^{-1} \mathbb{W}_{tot}(s) Q = \mathbb{W}_1((L+1)s)}$$

and the same relation thus holds for the large deviation functions

$$\boxed{\psi_{tot}(s) = \psi_1((L+1)s)}$$

This means that determining the statistics of the total current is equivalent to determining the statistics of the current to the right reservoir, which is in general more simple.

• Symmetry 2 : invariance by rotation

in periodic boundary condition : let's explicit the S^{xyz} operators :

$$\begin{cases} S^{\pm} = S^x \pm i S^y & \text{ie: } S^x = \frac{S^+ + S^-}{2} \quad S^y = \frac{S^+ - S^-}{2i} \\ \hat{n} = \frac{N}{2} + S^z & \\ \check{n} = \frac{N}{2} - S^z & \end{cases}$$

$p=q=1$

$$\mathbb{W} = \sum_{k=1}^{L} \left\{ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \check{n}_k \hat{n}_{k+1} \right\}$$

$-$ idem $k \to k+1$

$$= \sum_{k=1}^{L} \left\{ (S_k^x + i S_k^y)(S_{k+1}^x - i S_{k+1}^y) + (S_k^x - i S_k^y)(S_{k+1}^x + i S_{k+1}^y) - \left(\frac{N}{2} + S_k^z\right)\left(\frac{N}{2} - S_k^z\right) \right.$$

$$2\left\{ \underbrace{S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + S_k^z S_{k+1}^z}_{\text{denoted by } \vec{S}_k \cdot \vec{S}_{k+1}} + \text{cte} \right\}$$

$$\boxed{\mathbb{W} = 2 \sum_{k=1}^{L} \vec{S}_k \cdot \vec{S}_{k+1}} \qquad \vec{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix}$$

$\mathbb{R}$: matrix of rotation in 3D . $\mathbb{R}\vec{S}$ is again a set of 3 spin operator

Besides: $(\mathbb{R}\vec{S}_k) \cdot (\mathbb{R}\vec{S}_{k+1}) = \vec{S}_k \cdot \vec{S}_{k+1}$ (invariance by rotation of the scalar product)

One thus have an invariance by rotation of the spin operators $\vec{S}$

It does not affect the bulk part of $\mathbb{W}$ to apply $\mathbb{R}$.

. **Use** : for $\mathbb{W}_1(s)$ in open boundary conditions, $\mathbb{R}$ modifies these boundary terms.

After the rotation one can reinterpret them, for $s \neq 0$ as a system in contact with different densities (eg. at eq.)

$\rightarrow$ $\boxed{\text{Mapping Btw equilibrium and non-equilibrium}}$