4.1
Lecture on stoch processes

• Continuum Limit of Exclusion Processes

One now tackle the ASEP $\overset{p/N}{\underset{q/N}{\rightleftarrows}}$ (still in p.b.c)

$$\mathbb{W} = \frac{1}{N}\sum_{k=1}^{L}\left\{ p\, S^+_{k+1} S^-_k + q\, S^+_k S^-_{k+1} - p\, \check n_{k+1}\hat n_k - q\, \hat n_{k+1}\check n_k \right\}$$

$$S[\hat\rho,\rho] = \int_0^t d\tau \sum_{k=1}^L \left\{ \hat\rho_k \partial_\tau\rho_k + p\,\rho_k(1-\rho_{k+1})\big(\underbrace{1-e^{\hat\rho_{k+1}-\hat\rho_k}}_{\hat\rho_{k+1}-\hat\rho_k+\frac12(\hat\rho_{k+1}-\hat\rho_k)^2}\big) + q\,\rho_{k+1}(1-\rho_k)\big(\underbrace{1-e^{-(\hat\rho_{k+1}-\hat\rho_k)}}_{-(\hat\rho_{k+1}-\hat\rho_k)+\frac12(\hat\rho_{k+1}-\hat\rho_k)^2}\big)\right\}$$

One again performs a gradient expansion $\hat\rho_{k+1}-\hat\rho_k \ll 1$

$$= \int_0^t d\tau\sum_{k=1}^L\left\{\hat\rho_k\partial_\tau\rho_k + (\hat\rho_{k+1}-\hat\rho_k)\left[-p\rho_k(1-\rho_{k+1})+q\rho_{k+1}(1-\rho_k)\right] + \frac12(\hat\rho_{k+1}-\hat\rho_k)^2\underbrace{\left[\rho_k(1-\rho_{k+1})+\rho_{k+1}(1-\rho_k)\right]}_{=2\rho_k(1-\rho_k)+O(1/L)}\right\}$$

$\sigma(\rho)=2\rho(1-\rho)$

$$S[\hat\rho,\rho] = \int_0^t d\tau \sum_{k=1}^L\left\{\hat\rho_k\partial_\tau\rho_k + (\hat\rho_{k+1}-\hat\rho_k)\left[-p\rho_k+q\rho_{k+1} + (p-q)\rho_k\rho_{k+1}\right] + \frac12(\hat\rho_{k+1}-\hat\rho_k)^2\sigma(\rho_k)\right\}$$

× Case of the WASEP: $p = 1+\frac{E}{2}/L$ $\quad q = 1-\frac{E}{2}/L$ $\quad\rightarrow p-q = E/L = O(1/L)$

$$S[\hat\rho,\rho] = \int_0^{t_{mic}} d\tau_{mic}\sum_{k=1}^L\left\{\hat\rho_k\partial_{\tau_{mic}}\rho_k + (\hat\rho_{k+1}-\hat\rho_k)(\rho_{k+1}-\rho_k) + (\hat\rho_{k+1}-\hat\rho_k)\frac{E}{L}\big[\overbrace{\rho_k\rho_{k+1}-\tfrac12\rho_k-\tfrac12\rho_{k+1}}^{=\frac12(2\rho_k^2-\rho_k)=\frac{\sigma}{2}}\big] + \frac12(\hat\rho_{k+1}-\hat\rho_k)^2\sigma(\rho_k)\right\}$$

All terms are of order $1/L^2$: one sets $t=L^2 t_{mic}$ i.e. $\partial_{\tau_{mic}} = L^{-2}\partial_\tau$

$$\boxed{S[\hat\rho,\rho] = L\int_0^t d\tau\int_0^1 dx\left\{\hat\rho\,\partial_\tau\rho + \nabla\hat\rho\,\nabla\rho - E\,\nabla\hat\rho\,\sigma(\rho) - \frac12\sigma(\rho)(\nabla\hat\rho)^2\right\}}$$

The WASEP is thus having a diffusive behaviour

$$k \underset{q}{\overset{p}{\rightleftarrows}} k+1 \qquad \left| \begin{array}{l} p = 1+\varepsilon/2 \qquad q = 1-\varepsilon/2 \qquad q+p = 2 \\ p-q = \varepsilon \end{array} \right.$$

[illegible]

$$\mathbb{W} = \frac{1}{N}\sum_k p\, S^+_{k+1} S^-_k + q\, S^+_k S^-_{k+1} - p\, \check{n}_{k+1}\hat{n}_k - q\, \hat{n}_{k+1}\check{n}_k$$

$t \simeq t_{mic}$

$$S[\hat{\rho},\rho] = \int_0^t d\tau \sum_k \left\{ \hat{\rho}_k \partial_\tau \rho_k + p\,\rho_k(1-\rho_{k+1})\Big(\underbrace{1-e^{\hat{\rho}_{k+1}-\hat{\rho}_k}}_{-(\hat{\rho}_{k+1}-\hat{\rho}_k)-\frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2}\Big) + q\,\rho_{k+1}(1-\rho_k)\Big(\underbrace{1-e^{-(\hat{\rho}_{k+1}-\hat{\rho}_k)}}_{(\hat{\rho}_{k+1}-\hat{\rho}_k)+\frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2}\Big) \right\}$$

Gradient expansion: $\hat{\rho}_{k+1}-\hat{\rho}_k \ll 1$

$$= \int_0^t d\tau \sum_k \left\{ \hat{\rho}_k \partial_\tau \rho_k + (\hat{\rho}_{k+1}-\hat{\rho}_k)\Big[\underbrace{q\rho_{k+1}(1-\rho_k) - p\rho_k(1-\rho_{k+1})}_{\underbrace{\rho_{k+1}(1-\rho_k)-\rho_k(1-\rho_{k+1})}_{\rho_{k+1}-\rho_k} + \frac{\varepsilon}{2}\big[\underbrace{\rho_{k+1}(1-\rho_k)+\rho_k(1-\rho_{k+1})}_{\simeq\, \sigma(\rho_k)}\big]}\Big] - \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2 \overbrace{(p+q)\rho_k(1-\rho_k)}^{\sigma(\rho_k)} \right\}$$

$$\boxed{S[\hat{\rho},\rho] = \int_0^t d\tau \sum_k \left\{ \hat{\rho}_k \partial_\tau \rho_k + (\hat{\rho}_{k+1}-\hat{\rho}_k)(\rho_{k+1}-\rho_k) - \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2\sigma(\rho_k) + (\hat{\rho}_{k+1}-\hat{\rho}_k)\frac{\varepsilon}{2}\sigma(\rho_k) \right\}}$$

One now assume for simplicity that
$$\begin{cases} \rho_k = \overset{\text{mean density } \rho_0}{\frac{1}{2}} + \varphi_k, & \varphi_k \ll 1 \\ \hat{\rho}_k = 0 + \hat{\varphi}_k, & \hat{\varphi}_k \ll 1 \end{cases}$$

Then: $\sigma(\rho_k) = 2\left(\frac{1}{2}+\varphi_k\right)\left(\frac{1}{2}-\varphi_k\right) = \frac{1}{2} - 2\varphi_k^2$. In the action: constant term vanishes with $\sum \hat{\rho}_{k+1}-\hat{\rho}_k$.

$\sigma(\rho_0)$ dominant

$$S[\hat{\rho},\rho] = \int_0^{t_{mic}} d\tau \sum_k \left\{ \underbrace{\hat{\varphi}_k \partial_{\tau_{mic}} \varphi_k}_{\sim L^{\alpha+\hat{\alpha}-z} \;\sim L^{2\alpha-z}} + \underbrace{\underbrace{(\hat{\varphi}_{k+1}-\hat{\varphi}_k)(\varphi_{k+1}-\varphi_k)}_{\sim L^{\alpha+\hat{\alpha}-2}} - \underbrace{\frac{1}{4}(\hat{\varphi}_{k+1}-\hat{\varphi}_k)^2}_{\sim L^{2\hat{\alpha}-2}}}_{\Rightarrow\ \boxed{\alpha=\hat{\alpha}}} - \underbrace{\varepsilon(\hat{\varphi}_{k+1}-\hat{\varphi}_k)\varphi_k^2}_{\sim L^{3\alpha-1}} \right\}$$

$\varphi \sim L^{\alpha}\,\Phi$

$\hat{\varphi} \sim L^{\hat{\alpha}}\,\hat{\Phi}$

$\tau_{mic} \sim L^z\, t$

$\partial_{\tau_{mic}} \sim L^{-z}\,\partial_{\tau_{mic}}$

with $\alpha = \hat{\alpha} = -1/2$: $\sim L^{-3}$: subdominant

$\sim L^{\frac{1}{2}}$

$2\alpha - z = 3\alpha - 1$

$z = 1 - \underbrace{\alpha}$ with $\alpha = -1/2 \Rightarrow \boxed{z = 3/2}$ dynamical exponent

a proper argument for this arise from a symmetry

. Full argument: renormalization group

- Large deviations of the activity in SSEP: jamming

$\langle e^{-sK} \rangle$ with $s = \frac{\lambda}{L^2}$ $\qquad \rho = q = 1/N$

$$\mathbb{W}(s) = \frac{1}{N}\sum_k \left\{ S_k^+ S_{k+1}^- e^{-\frac{\lambda}{L^2}} + S_k^- S_{k+1}^+ e^{-\lambda/L^2} - \hat{n}\check{n} - \check{n}\hat{n} \right\}$$

$\nearrow -\frac{\lambda}{L^2}\left[\rho_k(1-\rho_{k+1}) + \rho_{k+1}(1-\rho_k)\right] = -\frac{\lambda}{L^2}\sigma(\rho_k)$

diffusive scaling remains.

$$S[\rho,\hat{\rho};s] = \int \sum_k \left\{ \hat{\rho}_k \partial_t \rho_k + \overbrace{(\hat{\rho}_{k+1}-\hat{\rho}_k)(\rho_k-\rho_{k+1})}^{1/L^2} - \overbrace{\tfrac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2\sigma(\rho_k)}^{\sim 1/L^2} + \overbrace{\frac{\lambda}{L^2}\sigma(\rho_k)}^{1/L^2} \right\}$$

$t \to L^2 t$ (diffusive scaling

$$\boxed{S[\hat{\rho},\rho;s] = L\int_0^t d\tau \int_0^1 dx \left\{ \hat{\rho}\partial_\tau \rho + \nabla\rho\nabla\hat{\rho} - \frac{1}{2}\sigma(\rho)(\nabla\hat{\rho})^2 + \lambda\sigma(\rho) \right\}}$$

× Simplest hypothesis: @ saddle, $\rho(x,t) = \rho(x) = \rho_0$ uniform

One obtains $\langle e^{-sK} \rangle \sim e^{-\lambda t L \sigma(\rho_0)}$ $\qquad t^{\text{micro}} = L^{-2} t^{\text{micro}}$ $\quad$ ie $\boxed{\psi(\lambda) = -L^{-1}\sigma(\rho_0)\lambda}$

$\psi(s) = -L\,\sigma(\rho_0)\,s$

Remark: $\frac{1}{t}\langle K \rangle = \left\langle \sum_k n_k(1-n_{k+1}) + n_{k+1}(1-n_k) \right\rangle = L\sigma(\rho_0)$ and $-\psi'(s) = L\sigma(\rho_0)$ OK.

× Question: is the uniform saddle point stable? (ie is it a correct saddle?)

Solution expand around the saddle

$\rho(x,t) = \rho_0 + \varphi(x,t) \qquad \varphi \ll \rho$

$\hat{\rho}(x,t) = 0 + \hat{\varphi}(x,t) \qquad \hat{\varphi}$ small

Then: $\sigma(\rho) = \sigma(\rho_0) - 2\varphi^2$

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$$S[\hat{\rho},\rho;\lambda] = \underbrace{S[\hat{\rho}_c,\rho_c;\lambda]}_{Lt\,\sigma(\rho_0)} + \underbrace{S[\hat{\varphi},\varphi;\lambda]}_{\text{fluctuations}}$$

$$S[\hat{\varphi},\varphi;\lambda] = L\int_0^t dt \int_0^1 dx \left\{ \hat{\varphi}\partial_t\varphi + \nabla\hat{\varphi}\nabla\varphi - \frac{1}{2}\overbrace{\sigma(\rho_0)}^{\equiv\sigma_0}(\nabla\hat{\varphi})^2 - 2\lambda\varphi^2 \right\}$$

Fourier transform

$$i\omega\hat{\varphi}\varphi + q^2\hat{\varphi}\varphi - \frac{1}{2}\sigma_0 q^2\hat{\varphi}^2 - 2\lambda\varphi^2 \text{ } \Big\} \to \text{diagonal}$$

$\omega \in \mathbb{R}$ continuous, $(t\to\infty)$ $\quad q = \frac{2\pi n}{L}$ discrete

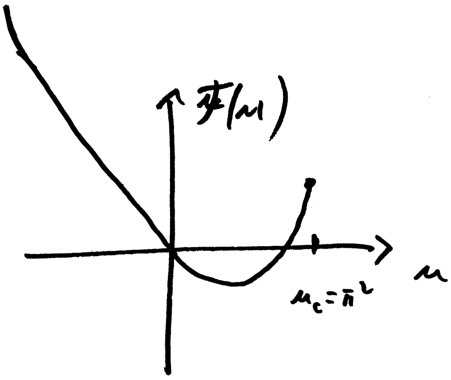
Takes a matrix form

$$\begin{pmatrix}\hat{\varphi}\\ \varphi\end{pmatrix} \Omega \begin{pmatrix}\hat{\varphi}\\ \varphi\end{pmatrix}$$

This a diagonal Gaussian integral; the result is:

$$\boxed{\psi_K(s) = -\sigma(\rho_0)Ls + \underbrace{\frac{2\pi^2}{L^2}\sum_{n\in\mathbb{Z}}\left\{ n^2 - \sqrt{n^2(n^2 - L^2 s\sigma(\rho_0)/\pi^2)} - L^2\frac{1}{\pi}s\frac{1}{2}\sigma(\rho_0)\right\}}}$$

universal funct $\mathcal{F}(u)$ $\quad u = sL^2\sigma(\rho_0)$



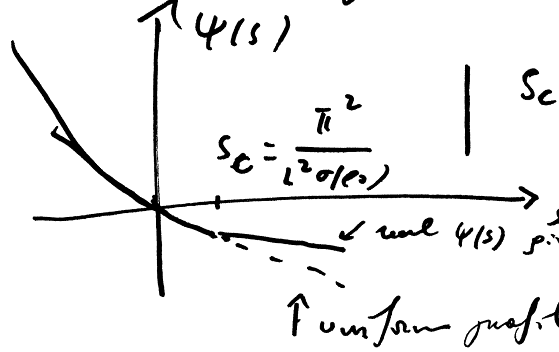
$$\mathcal{F}(u) = \sum_{n\in\mathbb{Z}}\left\{ n^2 - \sqrt{n^2(n^2 - u/\pi^2)} - \frac{1}{2}\frac{u}{\pi^2}\right\}$$

indeed: | we see that there is a problem in $u = \pi^2$

$\mathcal{F}(u)$ defined only for $u < u_c = \pi^2$

This means that for s large enough: $\quad s > \frac{\pi^2}{L^2\sigma(\rho_0)}$

the steady uniform profile $\rho(x,t) = \rho_0$ is unstable.



| s_c is a dynamical critical point. presenting jamming

$\leftarrow$ real $\psi(s)$ given by $\rho(x,t)$ non-uniform

$\uparrow$ uniform profile

• Dynamical Phase Transition in KCM.

1d F-A model : $0 \leq n_i \leq N$ occupation number

$1 \quad \cdots \quad L$

annihil/creation at rate $\frac{1-c}{N}$ / $\frac{c}{N}$ proportional to # neighbors

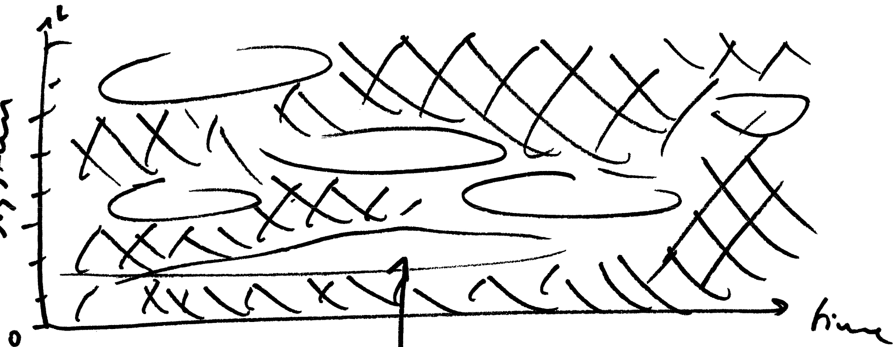
$$\begin{aligned} W(n_i \to n_i+1) &= \frac{c}{N}(N-n_i)\,(n_{i-1}+n_{i+1}) \\ W(n_i \to n_i-1) &= \frac{1-c}{N}\, n_i \,(n_{i-1}+n_{i+1}) \end{aligned}$$

choice — kinetic constraint

$\langle n_i \rangle = Nc$

Eq. distribution identical as when no kinetic constraint : $P(n_i) = \binom{N}{n_i}(cN)^{n_i}((1-c)N)^{N-n_i}$

Dynamics is very different!



bubbles of activity – similar to a in space-time (static) phase coexistence @ $T=T_c$ in 1st order phase transition

Mean field approach : 1 sites , n = # of active neighbors

$$\boxed{\begin{aligned} W(n \to n+1) &= \frac{c}{N}\,\overbrace{(N-n)}^{\text{choice of the void}}\; n \\ W(n \to n-1) &= \frac{1-c}{N}\,\underbrace{n}_{\text{choice of the active particle}}\,(n-1) \end{aligned}}$$

n ← # neighbors of the void

$(n-1)$ ← # neighbors of the particle

$$\boxed{W(s) = \frac{1}{N}\left[c\left(e^{-s}-(N-\hat{n})\right)\hat{n} + (1-c)\left(e^{-s}-\hat{n}\right)(\hat{n}-1)\right]}$$

$s \leftrightarrow K$

Action : in the limit $N \to \infty$ (large system size)

$$\boxed{S_s[\hat\rho,\rho] = \int_0^t d\tau \left\{ \hat\rho\, \partial_\tau \rho + \underbrace{c\rho(1-\rho)\left(-e^{-s}e^{\hat\rho}+1\right) + (1-c)\rho^2\left(-e^{-s}e^{-\hat\rho}+1\right)}_{\mathcal{H}} \right\}}$$

One searches for a steady saddle point $\frac{\delta S}{\delta \rho} = \frac{\delta S}{\delta \hat\rho} = 0$ i.e.

$$0 = \frac{\partial \mathcal{H}}{\partial \rho} = -c2\rho\left(1-e^{-s}e^{\hat\rho}\right) + 2(1-c)\rho\left(1-e^{-s}e^{-\hat\rho}\right)$$

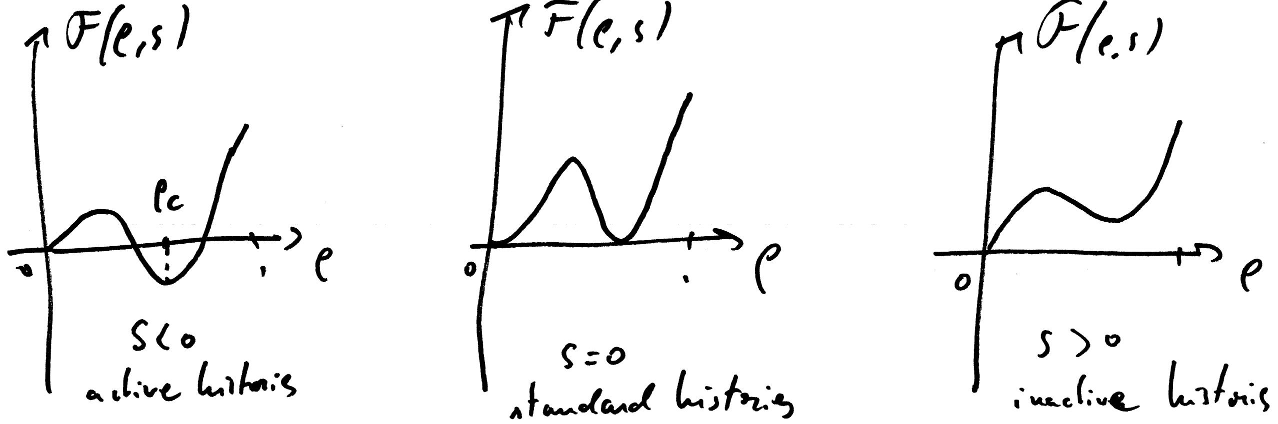
$$0 = \frac{\partial \mathcal{H}}{\partial \hat\rho} = -c\rho(1-\rho)e^{-s}e^{\hat\rho} + (1-c)\rho^2 e^{-s}e^{-\hat\rho} \Rightarrow \boxed{e^{\hat\rho} = e^{-s}\sqrt{\frac{1-c}{c}\frac{\rho}{1-\rho}}}$$

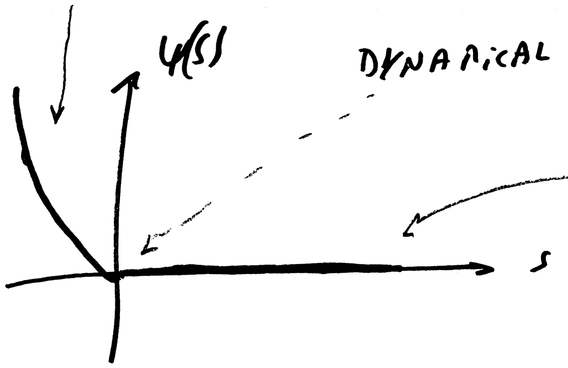
Instead of solving the full equations one changes $\min S_s[\hat\rho,\rho]$ into

$$\boxed{\Psi(s) = -\min_\rho \mathcal{F}(\rho,s)} \qquad \text{with } \mathcal{F}(\rho,s) = S_s[\hat\rho,\rho]$$

$\mathcal{F}(\rho,s)$ is a "dynamical Landau free energy"

$$\boxed{\mathcal{F}(\rho,s) = \dots = c\rho(1-\rho) + (1-c)\rho^2 - 2e^{-s}\sqrt{c(1-c)\rho(1-\rho)}}$$





DYNAMICAL PHASE COEXISTENCE

Competition btw active and inactive histories

Corresponding to the "bubbles" of inactivity in space-time

- **Numerical determination of $\psi(s)$: Cloning Algo.**

see K

$$\left(\mathbb{W}(s)\right)_{\mathcal{C}\mathcal{C}'} = \underbrace{e^{-s}\, W(\mathcal{C}'\to\mathcal{C}) \quad - r(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}}_{\text{non-probability conserving}}$$

$r(\mathcal{C}) = \sum_{\mathcal{C}'} W(\mathcal{C}\to\mathcal{C}')$

$W_s(\mathcal{C}\to\mathcal{C}') = e^{-s}\, W(\mathcal{C}\to\mathcal{C}')$

$r_s(\mathcal{C}) = \sum_{\mathcal{C}'} W_s(\mathcal{C}\to\mathcal{C}')$

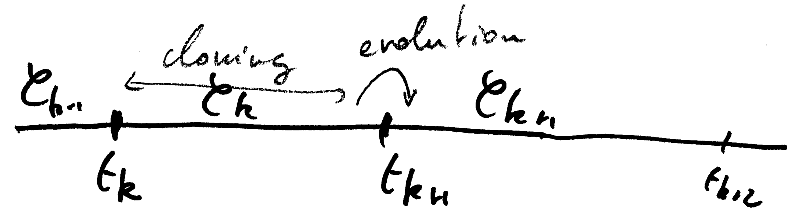
Interpretation in terms of s-modified dynamics + cloning:

$$\left(\mathbb{W}(s)\right)_{\mathcal{C}\mathcal{C}'} = \underbrace{W_s(\mathcal{C}'\to\mathcal{C}) - r_s(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}}_{\substack{\text{s-modified dynamics}\\ \text{with } W_s(\mathcal{C}\to\mathcal{C}') \text{ as rates}}} \;+\; \underbrace{\delta r_s(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}}_{\substack{\text{cloning part.}\\ \text{(rate of cloning)}}}$$

$\delta r_s(\mathcal{C}) = r_s(\mathcal{C}) - r(\mathcal{C})$

Evolution : N copies of the system (N large ~ 1000)

× each copy evolves with rates $W_s(\mathcal{C}\to\mathcal{C}')$ in continuous time



for a single copy

. t_{k+1} is found (1st copy to evolve, ie with minimal t_{k+1})
. $\mathcal{C}_{k+1}$ is chosen with probability $W_s(\mathcal{C}_k\to\mathcal{C}_{k+1})/r_s(\mathcal{C}_k)$
. t_{k+2} is drawn from exponential prob $\quad r_s(\mathcal{C}_{k+1})\, e^{-(t_{k+2}-t_{k+1})\, r_s(\mathcal{C}_{k+1})}$

ie $\Delta t = t_{k+2} - t_{k+1}$ ———— $r_s(\mathcal{C}_{k+1})\, e^{-\Delta t\, r_s(\mathcal{C}_{k+1})}$

× each copy is cloned; here, on $[t_k, t_{k+1}]$:

the copy should have $\quad Y = e^{-(t_{k+1}-t_k)\,\delta r_s(\mathcal{C}_k)}$ offsprings on $[t_k, t_{k+1}]$ (including itself)

Y is non-integer. One defines $y = \lfloor Y + \text{random}(0,1) \rfloor$

uniform real $\in [0,1[$

↑ lower integer part

the copy should have y (integer) offsprings (including itself)

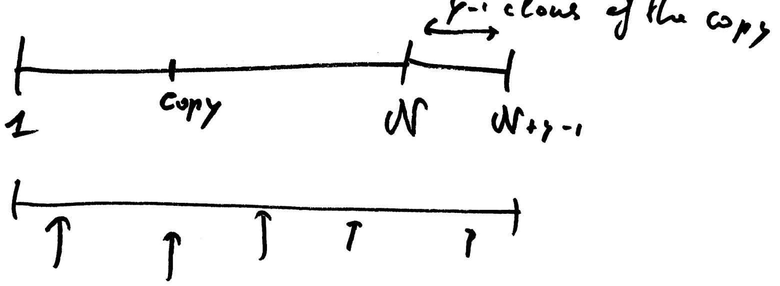
But one keeps the number of clones constant, keeping track of the cloning coefficients Z_p

Here are the rules:

- $y<1$: the copy should be killed.
  (i.e. $y=0$) . To keep $\mathcal{N}$ constant, one replaces the copy by one of the other $\mathcal{N}-1$ copies (chosen uniformly)
  . Then: $Z_p = \dfrac{\mathcal{N}-1}{\mathcal{N}}$

- $y=1$: nothing is done (no cloning)

- $y>1$: the copy should have $y-1$ clones of itself:



one then chooses uniformly on $[\![1, \mathcal{N}+y-1]\!]$ copies to kill.

$$Z_p = \frac{\mathcal{N}+y-1}{\mathcal{N}}$$

- In the large time limit: $\mathcal{N}(t) = Z_{p_{final}} \cdots Z_2 Z_1 \mathcal{N}(0) \sim e^{t\psi(s)}$

You can compute $\langle \sum \log Z_p \rangle$ which is $\sim t\,\psi(s)$ in the large time limit
↑ average over runs

i.e. the slope of $\langle \sum \log Z_p \rangle$ w.r.t. t is $\psi(s)$