Lecture 4: From operators to field theories

The cases of Doi-Peliti creation/annihilation operators & of Spin operators for exclusion processes

- **MOTIVATIONS:**

× Quantum Mechanics: the wave function of being in x at time t, having started from a state ψ_0 at time 0 is

$$\psi(x,t) = \langle x | e^{i\hbar t H} | \psi_0 \rangle$$ where H is the Schrödinger operator.

Feynman's approach to compute $\psi(x,t)$ is to rewrite this propagator-like expression as a integral over all the possible paths followed by the particle:

$$\psi(x,t) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x(\tau)\, e^{i\hbar\, \underbrace{S[x(\tau),t]}_{\text{action whose expression is deduced from } H}}$$

There are many ways to construct such a "path integral" form. One method which is particularly well adapted is the use of coherent states

× Stochastic processes: One has $|P(t)\rangle = e^{tW} |P_0\rangle$ and thus the mean value of an observable O depending on the final state $|P(t)\rangle$ is:

$$\langle O(t) \rangle = \langle - | O e^{tW} \underbrace{|P_0\rangle}_{\text{initial state}}$$

× In a way very similar to that of the quantum mechanics, one will write $\langle O(t)\rangle$ in terms of a path-integral form.

• (Bosonic) Coherent states

• **Remark**: the equilibrium solution of the birth & death process

$$A \underset{c}{\overset{1}{\rightleftarrows}} 0 \quad \text{of operator} \quad \mathbb{W} = a + ca^{\dagger} - \hat{n} - c$$

is the Poisson law of density c: $\quad P_{eq}(n) = e^{-c}\dfrac{c^n}{n!}$

. This is easily checked from the detailed balance on rates $\begin{cases} W(n \to n+1) = c \\ W(n \to n-1) = n \end{cases}$

. One can also check that $\mathbb{W}|P_{eq}\rangle = 0$:

Indeed: $a|P_{eq}\rangle = e^{-c}\sum_{n\geq 0} \frac{c^n}{n!} n|n-1\rangle = e^{-c} c \sum_{n\geq 0} \frac{c^{n-1}}{(n-1)!}|n-1\rangle \overset{n \to n+1}{=} e^{-c} c \sum_{n\geq 0} \frac{c^n}{n!}|n\rangle$

hence: $\boxed{a|P_{eq}\rangle = c|P_{eq}\rangle}$

and $a^{\dagger}|P_{eq}\rangle = e^{-c}\sum_{n\geq 0}\frac{c^n}{n!}|n+1\rangle \overset{n \to n-1}{=} \frac{1}{c} e^{-c}\sum_{n\geq 0} \frac{c^n}{(n-1)!}|n\rangle = \frac{\hat{n}}{c}|P_{eq}\rangle$ $\quad (n-1)! = \frac{1}{n} n!$

$$\boxed{a^{\dagger}|P_{eq}\rangle = \frac{\hat{n}}{c}|P_{eq}\rangle}$$

. **Finally**: $\mathbb{W}|P_{eq}\rangle = (a + ca^{\dagger} - \hat{n} - c)|P_{eq}\rangle = \left(c + c\frac{\hat{n}}{c} - \hat{n} - c\right)|P_{eq}\rangle = 0$

• **Coherent state**: one has thus seen that $a|P_{eq}\rangle = c|P_{eq}\rangle$: $|P_{eq}\rangle$ is a right eigenvector of the annihilation operator a.
This is a coherent state.

. **left eigenvector of $a^{\dagger}$**:

* action of $a^{\dagger}$ on $\langle n|$: $\langle n|a^{\dagger} = \sum_m \langle n|a^{\dagger}\underbrace{|m\rangle}_{\delta_{n,m+1}}\langle m| = \langle n-1|$ $\quad \Rightarrow m = n-1$: $\boxed{\langle n|a^{\dagger} = \langle n-1|}$ with $\langle 0|a^{\dagger} = 0$
* let's now compute $\langle P_{eq}|a^{\dagger}$:

$\langle P_{eq}|a^{\dagger} = e^{-c}\sum_{n\geq 0}\langle n-1|\frac{c^n}{n!} = e^{-c}\sum_{n\geq 0}\langle n|\frac{c^n}{n!}\frac{!}{n+1}$ $\rightarrow$ this is not a good eigenvector.

. **left and right coherent states**: $z \in \mathbb{C}$

$$\boxed{\begin{array}{l} |z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n\geq 0}\frac{z^n}{n!}|n\rangle \\ \hline \langle z| = e^{-\frac{1}{2}|z|^2}\sum_{n\geq 0}(z^*)^n\langle n| \end{array}}$$

one has $\quad a|z\rangle = z|z\rangle$

$\langle z|a^{\dagger} = z^*\langle z|$

- **Properties:**

× normalisation: $\langle z|z\rangle = e^{-|z|^2}\sum_{n,m}\frac{z^{*n}z^m}{m!}\underbrace{\langle n|m\rangle}_{\delta_{nm}} = 1 \qquad \boxed{\langle z|z\rangle = 1}$

× scalar product: $\langle z_1|z_2\rangle = e^{-\frac{1}{2}|z_1|^2-\frac{1}{2}|z_2|^2}\sum_{n\geq 0}\frac{(z_1^* z_2)^n}{n!} = e^{\underbrace{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1^*z_2}_{|z_1-z_2|^2+z_1^*z_2+z_1z_2^*}}$

$$\boxed{\langle z_1|z_2\rangle = \exp\left(-\frac{1}{2}|z_1-z_2|^2+\frac{1}{2}\underbrace{(z_1^*z_2-z_1z_2^*)}_{2\,\mathrm{Im}\, z_1^*z_2}\right)}$$

× representat° of the identity: (central formula for the construction of path $\int$)

. lemma: $\left\{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}\, z^{*n} z^m e^{-\frac{1}{2}|z|^2} = n!\,\delta_{nm}\right\}$ $\quad \int = \int_0^{2\pi} d\theta\cdot\int_0^{+\infty} dr\, r\, r^{2n} e^{-\frac{1}{2}r^2} = \frac{1}{2}n!\,2\pi$

δ is obtained by angular integration

. hence: $\int_{\mathbb{C}}\frac{dx\,dy}{\pi}|z\rangle\langle z| = \int_{\mathbb{C}}\frac{dx\,dy}{\pi}\sum_{n,m}\frac{(z^*)^n z^m}{m!}e^{-|z|^2}|n\rangle\langle m| = \sum_{n\geq 0}\frac{n!}{n!}|n\rangle\langle n|$

$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}|z\rangle\langle z| = \mathbb{1}}$$

$\mathbb{1} = \sum_n |n\rangle\langle n|$ is the identity in the Fock space ($|n\rangle$)

× representation of the Fock vectors:

$$\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}e^{-\frac{1}{2}|z|^2}|z\rangle = \sum_{m\geq 0}\overbrace{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}z^m\frac{e^{-|z|^2}}{m!}}^{\frac{n!}{m!}\delta_{nm}=\delta_{nm}}|m\rangle = |n\rangle$$

thus
$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}z^{*n}e^{-\frac{1}{2}|z|^2}|z\rangle = |n\rangle}$$
and similarly
$$\boxed{\int_{\mathbb{C}}\frac{dx\,dy}{\pi}\frac{z^n}{n!}e^{-\frac{1}{2}|z|^2}\langle z| = \langle n|}$$

· Construction of the path integral : (n may be a vector $\vec{n}$)

× Let's consider a system starting from distribution P_0 at time 0 and evolving with operator $\mathbb{W}$ (or $\mathbb{W}(s)$ when considering l.d.f.)

The average of an observable $O(n)$ writes, at time t : $\langle O \rangle = \sum_n O(n) P(n,t)$

$$\langle O \rangle = \langle - | O(\hat{n}) | P(t) \rangle = \langle - | O(\hat{n}) \, e^{t\mathbb{W}} | P_0 \rangle$$

× Let's decompose $[0,t]$ into $\mathcal{N}$ steps $dt = \frac{t}{\mathcal{N}}$ and insert $\mathcal{N}+1$ representations of the identity $\mathbb{1} = \int_{\mathbb{C}} \frac{d^2 z_p}{\pi} |z_p\rangle\langle z_p|$ $\left(d^2 z_p \equiv d\mathrm{Re}\, z_p \, d\mathrm{Im}\, z_p \right)_{0 \le p \le \mathcal{N}}$ in the (exact relation) $e^{t\mathbb{W}} = e^{\mathcal{N} dt \mathbb{W}} = e^{dt\mathbb{W}} \cdots\cdots e^{dt\mathbb{W}}$ ($\mathcal{N}$ factors)

$$\langle O \rangle = \langle - | \overset{\mathbb{1}}{\downarrow} O(\hat{n}) \overset{\mathbb{1}}{\downarrow} e^{dt\mathbb{W}} \overset{\mathbb{1}\cdots}{\swarrow} \cdots\cdots \overset{\mathbb{1}}{\downarrow} e^{dt\mathbb{W}} \overset{\mathbb{1}}{\swarrow} | P_0 \rangle$$

$$= \int_{\mathbb{C}^{\mathcal{N}+1}} \frac{d^2 z_0}{\pi} \cdots \frac{d^2 z_{\mathcal{N}}}{\pi} \langle - | O(\hat{n}) | z_{\mathcal{N}} \rangle \langle z_{\mathcal{N}} | e^{dt\mathbb{W}} | z_{\mathcal{N}-1} \rangle \cdots\cdots \langle z_1 | e^{dt\mathbb{W}} | z_0 \rangle \langle z_0 | P_0 \rangle$$

× One has changed the product of $\mathcal{N}$ operators into a product of $\mathcal{N}$ numbers $\langle z_p | e^{dt\mathbb{W}} | z_{p-1} \rangle$

× Bulk terms : let's "normal order" $\mathbb{W}$ (all $a^\dagger$'s on the left, all a's on the right) using the commutation relation on the a & $a^\dagger$'s.

One defines the $\mathbb{C}$-valued function $W(z_2^*, z_1)$ as $\mathbb{W}$ where $\begin{cases} a^\dagger \text{ is replaced by } z_2^* \\ a \text{ is replaced by } z_1 \end{cases}$

Then : in the limit $dt \to 0$: $\left[\begin{array}{l} \text{using that } |z\rangle (\langle z|) \text{ is a} \\ \text{right (left) e.v. of } a \, (a^\dagger) \end{array} \right] \to W(z_{p+1}^*, z_p)$

$$\langle z_{p+1} | e^{dt\mathbb{W}} | z_p \rangle = \langle z_{p+1} | 1 + dt\,\mathbb{W} | z_p \rangle = \langle z_{p+1} | z_p \rangle + dt \, \langle z_{p+1} | \mathbb{W} | z_p \rangle + O(dt^2)$$

$$= \langle z_{p+1} | z_p \rangle \left(1 + dt \frac{W(z_{p+1}^*, z_p)}{\langle z_{p+1} | z_p \rangle} \right) + O(dt^2)$$

$$\boxed{\langle z_{p+1} | e^{dt\mathbb{W}} | z_p \rangle = \langle z_{p+1} | z_p \rangle \exp\left(dt \frac{W(z_{p+1}^*, z_p)}{\langle z_{p+1} | z_p \rangle} \right) + O(dt^2)}$$

× Boundary terms : One defines $O(z_{\mathcal{N}}) = \frac{\langle - | O(\hat{n}) | z_{\mathcal{N}} \rangle}{\langle - |}$; $\langle - | a^\dagger = \langle - |$; $O(z_{\mathcal{N}}) = O(\hat{n})$ with $\begin{cases} a^\dagger \text{ to } 1 \\ a \text{ to } z_{\mathcal{N}} \end{cases}$ and $P(z_0) = \langle z_0 | P_0 \rangle$

Besides : $\langle - | = e^{+\frac{1}{2}} \langle \underline{1} |$ (coherent state) and thus $\langle - | O(\hat{n}) | z_{\mathcal{N}} \rangle = e^{\frac{1}{2}} e^{-\frac{1}{2} - \frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}} O(z_{\mathcal{N}}) = e^{-\frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}} O(z_{\mathcal{N}})$

- Continuous time limit $\mathcal{N} \to \infty$ (ie $dt \to 0$):

One assumes that, in the integrals, the values of z_k's which dominate are such that

$$z_{p+1} - z_p = \mathcal{O}(dt)$$

. Then, if W is regular enough (it is a polynomial in general)

$$\frac{W(z^*_{p+1}, z_p)}{\langle z_{p+1} | z_p \rangle} = W(z^*_p, z_p) + \mathcal{O}(dt)$$

. Besides, one wants to write $\langle z_{p+1} | z_p \rangle$ as $e^{dt\ldots}$, or, better, exactly:

$$\langle z_{p+1} | z_p \rangle = \exp\Big(-\frac{1}{2} \underbrace{|z_{p+1} - z_p|^2}_{(z^*_{p+1} - z^*_p)(z_{p+1} - z_p)} + \frac{1}{2} z^*_{p+1} z_p - \frac{1}{2} z^*_p z_{p+1} \Big)$$

$$= \exp\Big(+ \frac{1}{2} z^*_{p+1} z_{p+1} - \frac{1}{2} z^*_p z_p + z^*_{p+1} z_p - z^*_{p+1} z_{p+1} \Big)$$

$$\boxed{\langle z_{p+1} | z_p \rangle = \exp\Big[\underbrace{\frac{1}{2}\big(|z_{p+1}|^2 - |z_p|^2\big)}_{\text{useful when summing: this becomes a telescopic sum.}} - z^*_{p+1}(z_{p+1} - z_p) \Big]}$$

- Gathering all steps:

$$\langle \mathcal{O} \rangle = \int \frac{d^2 z_0}{\pi} \cdots \frac{d^2 z_{\mathcal{N}}}{\pi} \exp\Big(\sum_{p=0}^{\mathcal{N}-1} \Big[\overbrace{\underbrace{\frac{1}{2}|z_{p+1}|^2 - \frac{1}{2}|z_p|^2}_{\frac{1}{2}|z_{\mathcal{N}}|^2 - \frac{1}{2}|z_0|^2} - z^*_{p+1}(z_{p+1} - z_p)}^{\text{exact}} + \overbrace{dt\, W(z^*_p, z_p) + \mathcal{O}(dt^2)}^{\text{approximate (but exact as } dt \to 0)} \Big] \Big)$$

$$\times P(z_0)\, \mathcal{O}(z_{\mathcal{N}}) \times e^{-\frac{1}{2}|z_{\mathcal{N}}|^2 + z_{\mathcal{N}}}$$

- Finally: denoting $z_p = \varphi(p\,dt)$, $z^*_p = \bar{\varphi}(p\,dt)$, $\int \prod_k \frac{d^2}{\pi} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}$, $z_{p+1} - z_p = dt\, \partial_t \varphi$, $\sum dt = \int dt$

One finds the 'path integral expression':

$$\boxed{\begin{aligned} \langle \mathcal{O} \rangle &= \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}\; \mathcal{O}(\varphi(t))\, P_0(\varphi(0))\, e^{-S[\hat{\varphi}, \hat{\varphi}]} \\ S[\hat{\varphi}, \hat{\varphi}] &= -\varphi(t) + \frac{1}{2}\varphi^2(0) + \int_0^t d\tau \Big[\bar{\varphi}\, \partial_\tau \varphi - W(\bar{\varphi}, \varphi) \Big] \end{aligned}}$$

• Example to check that everything works: $A \underset{c}{\overset{1}{\rightleftarrows}} 0$

* Let's determine the mean value of n at final time, with s associated to the activity K

One has $\partial_t P(n,s,t) = e^{-s}[cP(n-1,s,t) + (n+1)P(n+1,s,t)] - (n+c)P(n,s,t)$

or directly $\boxed{\mathbb{W}(s) = e^{-s}(ca^{+} + a) - (c+\hat{n})}$

ρ may depend on s

Let's search an eigenstate of the Poisson form: $|P(s)\rangle = e^{-\rho(s)} \sum_{n\geq 0} \frac{\rho(s)^n}{n!}|n\rangle$

One has: $\mathbb{W}(s)|P(s)\rangle = \left(e^{-s}\left(\frac{c}{\rho}\hat{n} + \rho\right) - (c+\hat{n})\right)|P(s)\rangle$ cf: $\left| \begin{array}{l} a|\rho\rangle = \rho|\rho\rangle \\ a^{+}|\rho\rangle = \frac{1}{\rho}\hat{n}|\rho\rangle \end{array}\right.$

$\left(e^{-s}\frac{c}{\rho} - 1\right)\hat{n} = 0$ if $\boxed{\rho = ce^{-s}}$

in that case

$= c(e^{-2s} - 1)\, P(s)$

Remark:

One also has

$\mathbb{W}(s) = (a^{+} - e^{-s})(a - e^{-s}c) + c(e^{-2s} - 1)$

There exists thus Q such that

$Q^{-1}\mathbb{W}(s)Q = a^{+}a + c(e^{-2s} - 1)$ ⇒ One thus has the full spectrum

In other words, [since $\psi(0) = 0$, this is indeed the max eigenv.]

$\boxed{|P(s)\rangle \text{ Poissonian of density } \rho = ce^{-s} \text{ is the eigenvector corresponding to } \psi(s) = c(e^{-2s} - 1)}$

* In the path integral formalism: $\langle \hat{n} \rangle_s = \dfrac{\langle \hat{n} e^{-sK}\rangle}{\langle e^{-sK}\rangle}$ with $\langle \hat{n} e^{-sK}\rangle = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \dots$

with $\mathcal{O}(\varphi) = \varphi$, $W_s(\bar{\varphi},\varphi) = e^{-s}(c\bar{\varphi} + \varphi) - (c + \bar{\varphi}\varphi)$

Steady-state saddle point equations $\frac{\delta S}{\delta \varphi} = 0 = \frac{\delta S}{\delta \bar{\varphi}}$ write:

$$\left\{\begin{array}{l} 0 = \dfrac{\partial W_s}{\partial \varphi} \\ 0 = \dfrac{\partial W}{\partial \bar{\varphi}} \end{array}\right. \quad \text{ie} \quad \left\{\begin{array}{l} 0 = e^{-s} - \bar{\varphi} \\ 0 = ce^{-s} - \varphi \end{array}\right. \quad \text{hence} \quad \boxed{\left\{\begin{array}{l} \bar{\varphi}_s = e^{-s} \\ \varphi_s = ce^{-s} \end{array}\right.}$$

the are steady-state saddle point solutions

Thus: $\psi = \frac{1}{t}S_\alpha[\varphi_s,\bar{\varphi}_s] = \frac{t}{t}W(\bar{\varphi}_s,\varphi_s) = e^{-s}(ce^{-s} + ce^{-s}) - (c + ce^{-2s}) = c(e^{-2s} - 1)$

$\underline{\psi(s) = c(e^{-2s} - 1) \text{ as expected}}$ and the mean density is $\underline{\varphi_s = ce^{-s}}$: α.

* Remark: for the density at intermediate time: $e^{\frac{t}{2}\mathbb{W}}\, \mathcal{O}(\hat{n})\, e^{\frac{t}{2}\mathbb{W}} \longleftrightarrow \langle z_p|\mathcal{O}|z_{p-1}\rangle$

One thus should replace $\mathcal{O}(z_N)$ by $\mathcal{O}^{\text{interm}}(\bar{z}_p, z_{p-1}) = \mathcal{O}^{\text{interm}}(\bar{z}_p, z_p) + O(dt)$ with $\left\{\begin{array}{l} a \leftrightarrow z_p \\ a^{+} \leftrightarrow \bar{z}_p \end{array}\right.$

In our case one finds $\boxed{\rho^{\text{interm}}(s) = \bar{\varphi}_s \varphi_s = ce^{-2s}}$ which is a less trivial result.

(4.7)

Lecture on Stoch processes 2012

- <u>Diffusion of particles; diffusive limit</u>:

We consider a lattice of L sites with particles diffusing symmetrically at rate 1

$$\mathbb{W} = \sum_{k=1}^{L} a_k^\dagger a_{k+1} + a_k a_{k+1}^\dagger - \hat{n}_k - \hat{n}_{k+1} = \sum_{k=1}^{L} -(a_{k+1}^\dagger - a_k^\dagger)(a_{k+1} - a_k) \quad \text{p.b.c } L+1 \equiv 1$$

The construction is similar as previously, with coherent states on each site.

$$\boxed{S[\hat{\varphi},\varphi] = \int_0^t d\tau \Big[\sum_k \big\{ \underbrace{(\hat{\varphi}_{k+1} - \hat{\varphi}_k)}_{\equiv \nabla_k \hat{\varphi}} \underbrace{(\varphi_{k+1} - \varphi_k)}_{\equiv \nabla_k \varphi} + \hat{\varphi}_k \partial_\tau \varphi_k \big\} \Big] + \sum_k \tfrac{1}{2} |\varphi_k^2(0) - \varphi_k(t)}$$

$$\underbrace{\int_0^t d\tau L^2 \underbrace{\int dx L}_{\Sigma}}_{\hookrightarrow L\int \Sigma} \; L^{-1}\partial_x \hat{\varphi} L^{-1} \partial_x \varphi$$

Besides: $\sum_k \nabla_k \hat{\varphi} \nabla_k \varphi = -\sum_k \hat{\varphi}_k \nabla_k \nabla_{k+1} \varphi_k$ (t)

Continuous-space limit: $x = ka = \frac{k}{L}$ $\quad \frac{1}{L} = a =$ lattice step $\quad \nabla_k \varphi \leftrightarrow \frac{\partial_x \varphi}{a^{-1}} = L^{-1} \partial_x \rho$

Diffusive scaling: the time that we have in (t) is microscopic: $t = t_{mic}$, $\tau = \tau_{mic}$

To go to the macroscopic scale, one sets: $t_{mic} = L^2 t$, $\tau_{mic} = L^2 \tau$ — $\boxed{\text{DIFFUSIVE SCALING}}$ — i.e.: $d\tau_{mic} = L^2 d\tau$, $\partial_{\tau_{mic}} = L^{-2} \partial_\tau$

Finally, in terms of the new field:

$$\boxed{S[\hat{\varphi},\varphi] = L \int_0^t d\tau \Big[\int_0^1 dx \big\{ \hat{\varphi} (\partial_t - \partial_x^2) \varphi \big\} \Big] + \int dx \tfrac{1}{2} \hat{\varphi}(0)\varphi(0) - \varphi(t)}$$

in what follows one forgets about boundary terms

- <u>Cole-Hopf transform</u>: One would like $\rho = \hat{\varphi}\varphi$ to play a role. It happens a correct representation is $\boxed{\begin{array}{c} \hat{\varphi} = e^{\hat{\rho}} \\ \varphi = \rho e^{-\hat{\rho}} \end{array}}$

$$\hat{\varphi} \partial_t \varphi = e^{\hat{\rho}} (\partial_t \rho - \rho \partial_t \hat{\rho}) e^{-\hat{\rho}} = \partial_t (\rho - \rho\hat{\rho}) + \hat{\rho} \partial_t \rho$$

$$\partial_x \hat{\varphi} \partial_x \varphi = e^{\hat{\rho}} (\partial_x \hat{\rho})(\partial_x \rho - \rho \partial_x \hat{\rho}) e^{-\hat{\rho}} = \partial_x \hat{\rho} \partial_x \rho - \rho (\partial_x \hat{\rho})^2.$$ Hence:

$$\boxed{S[\rho,\hat{\rho}] = L \int_0^t d\tau \int_0^1 dx \big\{ \hat{\rho} (\partial_t - \Delta) \rho + \rho (\partial_x \hat{\rho})^2 \big\}} + \text{boundary terms.}$$

• Simple case to train : without space.

One consider a field $f(t)$ satisfying

$$\underbrace{\partial_t f}_{\equiv \frac{f(t+dt)-f(t)}{dt}} = \underbrace{-V'(f)}_{\text{at time } t \text{ and not } t+dt} + \eta(t) \qquad \eta(t) \text{ white noise with } \langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$$

i.e. $P[\eta] \propto \exp\Big(-\frac{1}{2}\underbrace{\int_0^t d\tau \frac{\eta^2(\tau)}{2}}_{\int_0^t\int_0^t d\tau d\tau' \frac{\delta(\tau-\tau')}{2} f(\tau)f(\tau')}\Big)$

How can represent the probability of an history as $\langle \delta(\dots)\rangle$ and hence:

$$P[\eta] \propto \int_{i\mathbb{R}} \mathcal{D}\hat{f} \exp\Big(-\int_0^t \frac{d\tau}{2} 2\hat{f}^2(\tau) + \hat{f}(\tau)\eta(\tau)\Big)$$

$$\overbrace{\langle O(t)\rangle}^{\text{over histories of duration } t} = \int_{0<\tau<t} \mathcal{D}f(\tau)\mathcal{D}\eta(\tau)\, \overbrace{\underbrace{\delta\big(\partial_t f + V'(f) - \eta(t)\big)}}^{\text{impose the equation of Langevin}}\, P[\eta]\; P_0(f)$$

| the $\delta(\dots)$ is a 'product' over each time step of Dirac delta's on $f(t)$.

→ | so as to integrate over η, one transforms it into a product of Dirac delta's on η, step by step in time

⇒ | this induces a Jacobian, which is unity (or constant) thanks to the choice of Itō convention $\partial_t f \to \frac{f(t+dt)-f(t)}{dt}$

Integrating now on η, one finds

$$\boxed{\langle O(t)\rangle = \int \mathcal{D}f\mathcal{D}\hat{f}\, \exp\Big(-\int_0^t d\tau \Big[\hat{f}\cdot(\partial_t f + V'(f)) + \hat{f}^2\Big]\Big)\, O(f(t))\; P_0(f(0))}$$

- **Martin-Siggia-Rose formalism of Langevin Dynamics**

+ Janssen, de Dominicis

$x \in \mathbb{R}^d$ position of particles.

$\rho(x,t)$ density of particles.

Langevin equation for $\rho(x,t)$:

Again: Itô discretization

$$\partial_t \rho = -\partial_x \left(-\partial_x \rho + \xi \right)$$

with $\xi(x,t)$ white noise of variance

small noise $\langle \xi(x,t)\, \xi(x',t') \rangle = \frac{2}{L} \rho(x,t)\, \delta(x-x')\, \delta(t-t')$

i.e.: $P[\xi] \propto \exp\left[-\frac{1}{2} \int dx\, dt \, \frac{\xi^2(x,t)}{\frac{2}{L}\rho(x,t)} \right] \overset{e.g.}{\propto} \int_{i\mathbb{R}} \mathcal{D}\hat{\rho} \, \exp\left[-\int dx\, dt\, L\, \xi \partial_x \hat{\rho} + \rho (\partial_x \hat{\rho})^2 \right]$

- Average of an observable:

$$\langle \mathcal{O}(t) \rangle = \int \mathcal{D}\rho\, \mathcal{D}\xi \;\, \delta\big(\partial_t \rho + \partial_x(-\partial_x \rho + \xi) \big)\, P[\xi]\, P_0(\rho(0))$$

integrating over ξ

$$\boxed{\begin{gathered} \langle \mathcal{O}(t) \rangle = \int \mathcal{D}\rho\, \mathcal{D}\hat{\rho}\, \exp\left(-S[\hat{\rho},\rho] \right) P_0(\rho(0)) \quad , \text{ with the action} \\ S[\hat{\rho},\rho] = L \int_0^t d\tau \int dx \left(\hat{\rho}(\partial_t \rho - \partial_x^2 \rho) + \rho (\partial_x \hat{\rho})^2 \right) \end{gathered}}$$

This is the same action as previously

otherwise, one has to come back to phenomenological Langevin equations.

It gives, doing the steps, a construction of the mesoscopic Langevin equations

microscopic dynamics $\longrightarrow$ operator of evolution $\mathbb{W}$ [or $\mathbb{W}(s)$] $\longrightarrow$ Doi-Peliti construction $\langle \dots \rangle = \int \mathcal{D}\rho\, \mathcal{D}\hat{\rho}\, e^{-S[\rho,\hat{\rho}]}$ in discrete space

$\downarrow$ → determination of the correct diffusive scaling

$\langle \dots \rangle = \int \mathcal{D}\rho\, \mathcal{D}\hat{\rho}\, e^{-S[\rho,\hat{\rho}]}$ in continuous space $\longrightarrow$ Reading as a (possible) Langevin dynamics

- Counter-example:

$$A + A \xrightarrow{\gamma} 0$$

$$\mathbb{W} = \gamma\left(a^2 - \hat{n}(\hat{n}-1)\right)$$

normal order $a a^{\dagger} = a^{\dagger} a + \mathbb{1}$

$$a^{\dagger}a(a^{\dagger}a-1) = a^{\dagger}(a^{\dagger}a+1)a - a^{\dagger}a = a^{\dagger 2}a^2$$

$$\boxed{\mathbb{W} = \gamma(a^{\dagger 2} - 1)\,a^2}$$

thus

$$S[\hat{\varphi},\varphi] = \int_0^t dt\left(\hat{\varphi}\,\partial_t\varphi - \gamma(\hat{\varphi}^2 - 1)\varphi^2\right) \quad + \text{bound.}$$

One performs the similarity transformation $\hat{\varphi} \mapsto \hat{\varphi} - 1$

$$S[\hat{\varphi},\varphi] = \int_0^t dt\left(\hat{\varphi}\,\partial_t\varphi - \gamma(\hat{\varphi}^2 - 2\hat{\varphi})\varphi^2\right)$$

$$\hat{\varphi}(\partial_t + 2\gamma\varphi)\varphi \quad - \gamma\varphi^2\hat{\varphi}^2$$

$$\stackrel{?}{=} \int_0^t dt\ \hat{\varphi}\left(\partial_t + V'(\varphi)\right)\varphi \underset{\uparrow}{+} \gamma\,\underbrace{\sigma(\varphi)}_{>0}\,\hat{\varphi}^2$$

the sign corresponds to an "imaginary" noise

This means there is no Langevin (real noise) description of the reaction process $A + A \to 0$

The action is only well defined when one uses the

$\mathbb{W} \rightsquigarrow S[\varphi\hat{\varphi}]$ construction.

cfr: Peliti

lecture on Stoch-processes 2012 — 4.11

SU(2) COHERENT STATES :

- **Definition** : by analogy, one choses a Bernoulli-like form :

$$\boxed{|z\rangle = \frac{1}{(1+|z|^2)^{N/2}} \sum_{0\le n\le N} \binom{N}{n} z^n |n\rangle}$$

$$\boxed{\langle z| = \frac{1}{(1+|z|^2)^{N/2}} \sum_{0\le n\le N} z^{*n} \langle n|}$$

- **Properties** :
  - normalization : $\boxed{\langle z|z\rangle = 1}$
  - representation of the identity :

Non uniform measure

$$\boxed{\int_{\mathbb{C}} d\mu(z)\, |z\rangle\langle z| = \mathbb{1} \quad \text{with} \quad d\mu(z) = \frac{N+1}{\pi} \frac{d^2 z}{(1+|z|^2)^2}}$$

Exercise : check this representation using a determination of

$$\int_{\mathbb{C}} d\mu(z)\, \frac{z^n z^{*m}}{(1+|z|^2)^N}$$

  - mean values of operators : SU(2) coherent states are NOT eigenvectors

In the construction, in fact, one only needs

$$\boxed{\begin{aligned} \langle z|S^+|z\rangle &= N\frac{z^*}{1+|z|^2} \\ \langle z|S^-|z\rangle &= N\frac{z}{1+|z|^2} \\ \langle z|S^z|z\rangle &= \frac{1}{2}N\frac{|z|^2-1}{|z|^2+1} \end{aligned}}$$

or using

$$z = \frac{\rho}{1-\rho} e^{-\hat{\rho}}$$

$$\bar{z} = e^{\hat{\rho}}$$

$$\boxed{\begin{aligned} \langle z|S^+|z\rangle &= (1-\rho)e^{\hat{\rho}}N \\ \langle z|S^-|z\rangle &= \rho\, e^{-\hat{\rho}}N \\ \langle z|S^z|z\rangle &= (2\rho-1)N \\ \text{ie} \quad \langle z|\hat{n}|z\rangle &= \rho N \end{aligned}}$$

* remark : ⚠ since $|z\rangle$ is not an eigenvector of S^-

one do not has $\langle z| S^- S^- |z\rangle = \left(N \frac{z}{1+|z|^2}\right)^2$

contrary to the case of Doi-Peliti where $a^2|z\rangle = z^2|z\rangle$.

There are known formula to express the mean value $\langle z|S^-S^-|z\rangle$

(This encodes the finiteness of N) (see references)

- Path integral representation :

Following the same path as for Bosonic coherent space one arrives at,

$$\boxed{\begin{aligned} S[\hat{\rho},\rho] &= N\int_0^t dt\left[\hat{\rho}\,\partial_t\rho - W(\hat{\rho},\rho)\right] \\ W(\hat{\rho},\rho) &= \frac{1}{N}\langle z|\mathbb{W}|z\rangle \end{aligned}}$$

computed by the replacement rules

$S^+ \leftrightarrow (1-\rho)e^{+\hat{\rho}}$
$S^- \leftrightarrow \rho\, e^{-\hat{\rho}}$
$\hat{n} \leftrightarrow \rho$

Boundary terms need special care.

The same formula applies with s-modified dynamics so as to represent large deviation functions.

- Including several lattice sites : and keeping space discrete

$$S[\hat{\rho},\rho] = N\left\{\sum_{k=1}^{L}\int_0^t d\tau\left[\hat{\rho}_k\partial_t\rho_k\right] + \int_0^t d\tau\, W(\vec{\hat{\rho}},\vec{\rho})\right\}$$

- Case of Simple exclusion processes : the symmetric one

$$\mathbb{W} = \frac{1}{N} \sum_{k=1}^{L} \left\{ S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat n_k \check n_{k+1} - \hat n_{k+1} \check n_k \right\} \qquad \text{jump rates: } \frac{1}{N} = p = q$$

$$W(\vec{\hat\rho},\vec\rho) = \sum_{k=1}^{L} \Big[\rho_{k+1}(1-\rho_k) \underbrace{\left(e^{-(\hat\rho_{k+1}-\hat\rho_k)} - 1 \right)}_{\simeq -(\hat\rho_{k+1}-\hat\rho_k) + \frac12 (\hat\rho_{k+1}-\hat\rho_k)^2} + \rho_k (1-\rho_{k+1}) \underbrace{\left(e^{\hat\rho_{k+1}-\hat\rho_k} - 1 \right)}_{= (\hat\rho_{k+1}-\hat\rho_k) + \frac12 (\hat\rho_{k+1}-\hat\rho_k)^2} \Big]$$

This is the exact microscopic action.

We now perform a gradient expansion $(\hat\rho_{k+1} - \hat\rho_k \ll 1)$, yielding

$$W(\vec{\hat\rho},\vec\rho) = \sum_{k=1}^{L} -(\hat\rho_{k+1}-\hat\rho_k)\Big[\overbrace{\rho_{k+1}(1-\rho_k) - \rho_k(1-\rho_{k+1})}^{\rho_{k+1}-\rho_k} \Big] + \frac12 (\hat\rho_{k+1}-\hat\rho_k)^2 \Big[\overbrace{\rho_{k+1}(1-\rho_k) + \rho_k(1-\rho_{k+1})}^{\simeq 2\rho_k(1-\rho_k)} \Big]$$

$$\boxed{\sigma(\rho) = 2\rho(1-\rho)}$$

This imposes to take the diffusive scaling limit:

$$\rho_{k+1} - \rho_k \mapsto \frac1L \partial_x \rho \qquad dt \mapsto L^2 dt \qquad \sum_{k=1}^{L} \to L \int_0^1 dx$$

$$\hat\rho_{k+1} - \hat\rho_k \mapsto \frac1L \partial_x \hat\rho \qquad \partial_t \mapsto L^{-2} \partial_t$$

$$S[\hat\rho,\rho] = \int_0^t L^2 dt \left\{ \overbrace{\sum_{k=1}^{L}}^{L\int dx} \left[\hat\rho_k L^{-2} \partial_t \rho_k + \frac1L \partial_x \hat\rho \, \frac1L \partial_x \rho - \frac12 \sigma(\rho) L^{-2} (\partial_x \hat\rho)^2 \right] \right\}$$

$$\boxed{S[\hat\rho,\rho] = L \int_0^t dt \int_0^1 dx \left[\hat\rho \, \partial_t \rho + \partial_x \hat\rho \, \partial_x \rho - \frac12 \sigma(\rho) (\partial_x \rho)^2 \right]}$$

This action corresponds to a Langevin equation of the form

$$\partial_t \rho(x,t) = -\partial_x \left(-\partial_x \rho + \xi(x,t) \right), \qquad \langle \xi(x,t)\, \xi(x',t') \rangle = 2 \frac{\sigma(\rho)}{L} \delta(x-x') \delta(t-t')$$

which describes the SSEP in the fluctuating hydrodynamic limit.

• Another useful representation consists in setting a current $j(x,\tau)$ verifying the "continuity equation" (encoding conservation of particles)

$$\partial_t \rho + \partial_x j = 0 .$$

Then: $S[\hat{\rho},\rho] = L \int_0^t d\tau \int_0^1 dx \left(\partial_x \hat{\rho} \left(j + \partial_x \rho \right) + \frac{1}{2} \sigma(\rho) (\nabla \hat{\rho})^2 \right)$

Formally integrating over $\partial_x \hat{\rho}$ in $\int \mathcal{D}\hat{\rho}\, e^{-S[\rho,\hat{\rho}]}$ one find that

$$\boxed{\mathrm{Prob}\left[\rho(\tau)\right]_{\tau \le t} \asymp \exp\left(-\frac{1}{2L} \int_0^t d\tau \int_0^1 dx \, \frac{(j + \partial_x \rho)^2}{2\sigma(\rho)} \right)}$$

which is another representation of the fluctuating hydrodynamics

• Remark: The spatio-temporal noise has correlations

$$\langle \xi(x,t)\, \xi(x',t') \rangle = \frac{2}{L}\, \sigma(\rho)\, \delta(x-x')\, \delta(t-t') \qquad \sigma(\rho) = \sigma(\rho(x,t))$$

$$\sigma(\rho) = 2\rho(1-\rho)$$

thus when $\rho = 0$ (no particles)
or $\rho = 1$ (maximum density of particles)
the noise disappears, as physically expected.