Part V — DYNAMICAL PHASE TRANSITION: SOME EXAMPLES

① Exclusion Processes

One now tackle the ASEP $\underset{q/N}{\overset{p}{\rightleftarrows}}$ (still in p.b.c)

$$\mathbb{W} = \frac{1}{N}\sum_{k=1}^{L}\left\{ p\, S^+_{k+1} S^-_k + q\, S^+_k S^-_{k+1} - p\, \check{n}_{k+1}\hat{n}_k - q\, \hat{n}_{k+1}\check{n}_k \right\}$$

$$S[\hat\rho,\rho] = \int_0^t d\tau \sum_{k=1}^{L}\left\{ \hat\rho_k \partial_\tau \rho_k + p\, \rho_k(1-\rho_{k+1})\underbrace{\left(1-e^{\hat\rho_{k+1}-\hat\rho_k}\right)}_{\hat\rho_{k+1}-\hat\rho_k+\frac{1}{2}(\hat\rho_{k+1}-\hat\rho_k)^2} + q\,\rho_{k+1}(1-\rho_k)\underbrace{\left(1-e^{-\overset{\hat\rho_{k+1}-\hat\rho_k}{()}}\right)}_{-(\hat\rho_{k+1}-\hat\rho_k)+\frac{1}{2}(\hat\rho_{k+1}-\hat\rho_k)^2}\right.$$

One again performs a gradient expansion $\hat\rho_{k+1}-\hat\rho_k \ll 1$

$$= \int_0^t d\tau \sum_{k=1}^{L}\Big\{ \hat\rho_k \partial_\tau \rho_k + (\hat\rho_{k+1}-\hat\rho_k)\left[-p\rho_k(1-\rho_{k+1}) + q\rho_{k+1}(1-\rho_k)\right]$$
$$+\frac{1}{2}(\hat\rho_{k+1}-\hat\rho_k)^2 \underbrace{\left[\rho_k(1-\rho_{k+1}) + \rho_{k+1}(1-\rho_k)\right]}_{=2\rho_k(1-\rho_k)+O(1/L)}\Big\} \qquad \sigma(\rho)=2\rho(1-\rho)$$

$$S[\hat\rho,\rho] = \int_0^t d\tau \sum_{k=1}^{L}\Big\{ \hat\rho_k \partial_\tau \rho_k + (\hat\rho_{k+1}-\hat\rho_k)\left[-p\rho_k + q\rho_{k+1} + (p-q)\rho_k\rho_{k+1}\right]$$
$$+\frac{1}{2}(\hat\rho_{k+1}-\hat\rho_k)^2\,\sigma(\rho_k)\Big\}$$

× <u>Case of the WASEP</u>: $p = 1+\frac{E}{2L}$ $\quad q = 1-\frac{E}{2L}$ $\quad\rightarrow p-q = E/L = O(1/L)$

$$S[\hat\rho,\rho] = \int_0^{t_{mic}} d\tau_{mic}\sum_{k=1}^{L}\Big\{ \hat\rho_k \partial_{\tau_{mic}}\rho_k + (\hat\rho_{k+1}-\hat\rho_k)(\rho_{k+1}-\rho_k) + (\hat\rho_{k+1}-\hat\rho_k)\frac{E}{L}\overbrace{\left[\rho_k\rho_{k+1}-\frac{1}{2}\rho_k-\frac{1}{2}\rho_{k+1}\right]}^{=\frac{1}{2}(2\rho_k^2-\rho_k)=-\frac{\sigma}{2}}$$
$$+\frac{1}{2}(\hat\rho_{k+1}-\hat\rho_k)^2\,\sigma(\rho_k)\Big\}$$

All terms are of order $1/L^2$: one sets $t = L^2 t_{mic}$ i.e. $\partial_{\tau_{mic}} = L^{-2}\partial_\tau$

$$\boxed{S[\hat\rho,\rho] = L\int_0^t d\tau \int_0^1 dx \left\{ \hat\rho\,\partial_\tau\rho + \nabla\hat\rho\,\nabla\rho - E\,\nabla\hat\rho\,\sigma(\rho) - \frac{1}{2}\sigma(\rho)(\nabla\hat\rho)^2\right\}}$$

The WASEP is thus having a diffusive behaviour

$k \overset{p}{\underset{q}{\rightleftarrows}} k+1$

$$W = \frac{1}{N}\sum_k p\, S^+_{k+1} S^-_k + q\, S^+_k S^-_{k+1} - p\, \check{n}_{k+1}\hat{n}_k - q\, \hat{n}_{k+1}\check{n}_k \quad \Big| \quad \begin{array}{l} p = 1+\varepsilon/2 \\ p - q = \varepsilon \end{array} \qquad q = 1 - \varepsilon/2 \qquad q+p = 2$$

Lecture on Stochastic processes 2012

$t \equiv t_{mic}$

$$S[\hat{\rho},\rho] = \int_0^t d\tau \sum_k \Big\{ \hat{\rho}_k \partial_\tau \rho_k + p\,\rho_k(1-\rho_{k+1})\underbrace{\left(1 - e^{\hat{\rho}_{k+1}-\hat{\rho}_k}\right)} + q\,\rho_{k+1}(1-\rho_k)\underbrace{\left(1 - e^{-(\hat{\rho}_{k+1}-\hat{\rho}_k)}\right)}$$

Gradient expansion: $\hat{\rho}_{k+1} - \hat{\rho}_k \ll 1$ $\qquad -(\hat{\rho}_{k+1}-\hat{\rho}_k) - \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2 \qquad (\hat{\rho}_{k+1}-\hat{\rho}_k) + \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2$

$$= \int_0^t d\tau \sum_k \Big\{ \hat{\rho}_k \partial_\tau \rho_k + (\hat{\rho}_{k+1}-\hat{\rho}_k)\Big[\underbrace{q\rho_{k+1}(1-\rho_k) - p\rho_k(1-\rho_{k+1})}_{\rho_{k+1}(1-\rho_k) - \rho_k(1-\rho_{k+1}) + \frac{\varepsilon}{2}[\rho_{k+1}(1-\rho_k) + \rho_k(1-\rho_{k+1})]}\Big] - \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2 \overbrace{(p+q)\rho_k(1-\rho_k)}^{\sigma(\rho_k)}$$

$$\underbrace{\rho_{k+1}(1-\rho_k) - \rho_k(1-\rho_{k+1})}_{\rho_{k+1}-\rho_k} + \frac{\varepsilon}{2}\underbrace{[\rho_{k+1}(1-\rho_k) + \rho_k(1-\rho_{k+1})]}_{\simeq\, \sigma(\rho_k)}$$

$$\boxed{S[\hat{\rho},\rho] = \int_0^t d\tau \sum_k \Big\{ \hat{\rho}_k \partial_\tau \rho_k + (\hat{\rho}_{k+1}-\hat{\rho}_k)(\rho_{k+1}-\rho_k) - \frac{1}{2}(\hat{\rho}_{k+1}-\hat{\rho}_k)^2 \sigma(\rho_k) + (\hat{\rho}_{k+1}-\hat{\rho}_k)\frac{\varepsilon}{2}\sigma(\rho_k) \Big\}}$$

One now assumes for simplicity that $\begin{cases} \rho_k = \frac{1}{2} + \varphi_k, & \varphi_k \ll 1 \\ \hat{\rho}_k = 0 + \hat{\varphi}_k, & \hat{\varphi}_k \ll 1 \end{cases}$ (mean density ρ_0)

Then: $\sigma(\rho_k) = 2\left(\frac{1}{2}+\varphi_k\right)\left(\frac{1}{2}-\varphi_k\right) = \frac{1}{2} - 2\varphi_k^2$. In the action: constant terms vanish with $\sum \hat{\rho}_{k+1} - \hat{\rho}_k$.

$$S[\hat{\rho},\rho] = \int_0^{t_{mic}} d\tau \sum_k \Big\{ \underbrace{\hat{\varphi}_k \partial_{\tau_{mic}} \varphi_k}_{\sim L^{\alpha+\hat{\alpha}-z}} + \underbrace{\underbrace{(\hat{\varphi}_{k+1}-\hat{\varphi}_k)(\varphi_{k+1}-\varphi_k)}_{\sim L^{\alpha+\hat{\alpha}-2}} - \underbrace{\frac{1}{4}(\hat{\varphi}_{k+1}-\hat{\varphi}_k)^2}_{\sim L^{2\hat{\alpha}-2}}}_{\Rightarrow\ \boxed{\alpha = \hat{\alpha}}} - \underbrace{\varepsilon(\hat{\varphi}_{k+1}-\hat{\varphi}_k)\varphi_k^2}_{\sim L^{3\alpha-1}} \Big\}$$

$\sigma(\rho_0)$ dominant

$\varphi \sim L^{\alpha}\Phi$

$\hat{\varphi} \sim L^{\hat{\alpha}}\hat{\Phi}$

$\tau_{mic} \sim L^z t$

$\partial_{\tau_{mic}} \sim L^{-z}\partial_{\tau_{mic}}$

$\sim L^{2\alpha - z}$

with $\alpha = \hat{\alpha} = -1/2$: $\sim L^{-3}$: subdominant

$\sim L^{\frac{1}{2}}$

$2\alpha - z = 3\alpha - 1$

$z = 1 - \alpha$ with $\underbrace{\alpha = -1/2}$ $\Rightarrow$ $\boxed{z = 3/2}$ dynamical exponent

a proper argument for this arise from a symmetry

. Full argument: renormalization group

② Large deviations of the activity in SSEP: jamming

$\langle e^{-sK} \rangle$ with $s = \frac{\lambda}{L^2}$ $\rho = q = 1/N$

$$\mathbb{W}(s) = \frac{1}{N} \sum_k \left\{ S_k^+ S_{k+1}^- e^{-\frac{\lambda}{L^2}} + S_k^- S_{k+1}^+ e^{-\lambda/L^2} - \hat{n}\check{n} - \check{n}\hat{n} \right\}$$

$\nearrow -\frac{\lambda}{L^2}\left[\rho_k(1-\rho_{k+1}) + \rho_{k+1}(1-\rho_k)\right] = -\frac{\lambda}{L^2}\sigma(\rho_k)$

diffusive scaling remains.

$$S[\rho,\hat{\rho};s] = \int \sum_k \left\{ \hat{\rho}_k \partial_t \rho_k + \overbrace{(\hat{\rho}_{k+1}-\hat{\rho}_k)(\rho_k - \rho_{k+1})}^{1/L^2} - \frac{1}{2}\overbrace{(\hat{\rho}_{k+1}-\hat{\rho}_k)^2}^{\sim 1/L^2}\sigma(\rho_k) + \overbrace{\frac{\lambda}{L^2}\sigma(\rho_k)}^{1/L^2} \right\}$$

$t \mapsto L^2 t$ (diffusive scaling

$$\boxed{S[\hat{\rho},\rho;s] = L\int_0^t d\tau \int_0^1 dx \left\{ \hat{\rho}\partial_\tau \rho + \nabla\rho\nabla\hat{\rho} - \frac{1}{2}\sigma(\rho)(\nabla\hat{\rho})^2 + \lambda\,\sigma(\rho) \right\}}$$

× <u>Simplest hypothesis:</u> @ saddle, $\rho(x,t) = \rho(x) = \rho_0$ uniform

$t^{\text{macro}} = L^{-2} t^{\text{micro}}$

One obtains $\langle e^{-sK} \rangle \sim e^{-\lambda t L \sigma(\rho_0)}$ ie $\boxed{\psi(\lambda) = -L^{-1}\sigma(\rho_0)\lambda}$

$\psi(s) = -L\,\sigma(\rho_0)\,s$

Remark: $\frac{1}{t}\langle K \rangle = \left\langle \sum_k n_k(1-n_{k+1}) + n_{k+1}(1-n_k) \right\rangle = L\sigma(\rho_0)$ and $-\psi'(s) = L\sigma(\rho_0)$ OK.

× <u>Question:</u> is the uniform saddle point stable? (ie is it a correct saddle ?)

Solution expand around the saddle

$$\rho(x,t) = \rho_0 + \varphi(x,t) \qquad \varphi \ll \rho$$
$$\hat{\rho}(x,t) = 0 + \hat{\varphi}(x,t) \qquad \hat{\varphi} \text{ small}$$

Then: $\sigma(\rho) = \sigma(\rho_0) - 2\varphi^2$

Lecture on Stoch. Processes 2012

$$S[\hat{\rho},\rho;\lambda] = \underbrace{S[\hat{\rho}_c,\rho_c;\lambda]}_{Lt\,\sigma(\rho_0)} + \underbrace{S[\hat{\varphi},\varphi;\lambda]}_{\text{fluctuations}}$$

$$S[\hat{\varphi},\varphi;\lambda] = L\int_0^t dt \int_0^1 dx \left\{ \hat{\varphi}\partial_t\varphi + \nabla\hat{\varphi}\nabla\varphi - \frac{1}{2}\overbrace{\sigma(\rho_0)}^{=\sigma_0}(\nabla\hat{\varphi})^2 - 2\lambda\varphi^2 \right\}$$

Fourier transform

$$i\omega\hat{\varphi}\varphi + q^2\hat{\varphi}\varphi - \frac{1}{2}\sigma_0 q^2\hat{\varphi}^2 - 2\lambda\varphi^2 \Big\} \to \text{diagonal}$$

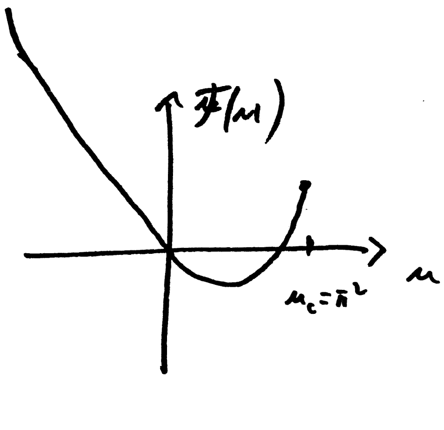
$\omega \in \mathbb{R}$ continuum, $(t\to\infty)$ $\quad q = \frac{2\pi n}{L}$ discrete

Takes a matrix form

$$\begin{pmatrix}\hat{\varphi}\\ \varphi\end{pmatrix}\Omega\begin{pmatrix}\hat{\varphi}\\ \varphi\end{pmatrix}$$

This a diagonal Gaussian integral; the result is:

$$\boxed{\Psi_k(s) = -\sigma(\rho_0)Ls + \underbrace{\frac{2\pi^2}{L^2}\sum_{n\in\mathbb{Z}}\left\{n^2 - \sqrt{n^2(n^2 - L^2 s\,\sigma(\rho_0)/\pi^2)} - L^2\frac{1}{\pi}s\frac{1}{2}\sigma(\rho_0)\right\}}}$$



universal funct $\mathcal{F}(u)$ $\quad u = sL^2\sigma(\rho_0)$

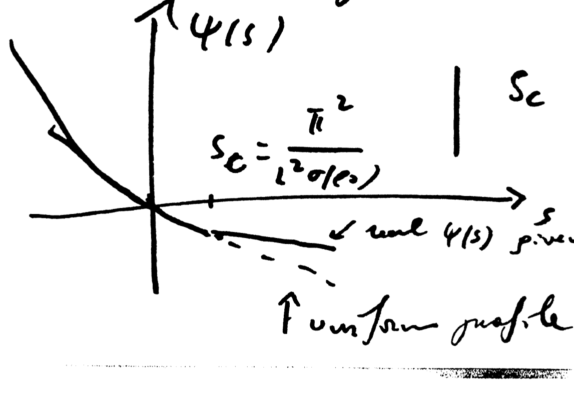
$$\mathcal{F}(u) = \sum_{n\in\mathbb{Z}}\left\{n^2 - \sqrt{n^2(n^2 - u/\pi^2)} - \frac{1}{2}\frac{u}{\pi}\right\}$$

indeed: | we see that there is a problem in $u = \pi^2$

<u>$\mathcal{F}(u)$ defined only for $u < u_c = \pi^2$</u>

This means that for s large enough: $\quad s > \frac{\pi^2}{L^2\sigma(\rho_0)}$

the steady uniform profile $\rho(x,t) = \rho_0$ <u>is unstable</u>.



| s_c is a dynamical critical point. presenting jamming

• Dynamical Phase Transition in KCM

1d F-A model : $0 \le n_i \le N$ occupation number

annihil / creation at rate $\frac{1-c}{N}$ / c/N proportional to # neighbors

$$\left|\begin{array}{l} W(n_i \to n_i+1) = \frac{c}{N}(N-n_i)(n_{i-1}+n_{i+1}) \\ W(n_i \to n_i-1) = \frac{1-c}{N}\, n_i\, (n_{i-1}+n_{i+1}) \end{array}\right.$$

↑ # choice ↑ kinetic constraint

$\langle n_i \rangle = Nc$

Eq. distribution identical as when no kinetic constraint : $P(n_i) = \binom{N}{n_i}(cN)^{n_i}((1-c)N)^{N-n_i}$

Dynamics is very different :

bubbles of activity – similar to a in space-time (static) phase coexistence @ $T=T_c$ in 1st order phase transition

Mean field approach : 1 sites , $n =$ # of active neighbors

$$\boxed{\begin{array}{l} W(n \to n+1) = \frac{c}{N}(N-n)\, n \\ W(n \to n-1) = \frac{1-c}{N}(n\,(n-1)) \end{array}}$$

choice of the void — # neighbors of the void

← # neighbors of the particle

↑ choice of the active particle

$$\boxed{\Delta W(s) = \frac{1}{N}\left[c\left(e^{-s}\widehat{S}^{+} - (N-\hat{n})\right)\hat{n} + (1-c)\left(e^{s}\widehat{S}^{-} - \hat{n}\right)(\hat{n}-1)\right]}$$

$s \leftrightarrow K$

Action: in the limit $N \to \infty$ (large system size)

$$\boxed{S_s[\hat{\rho},\rho] = \int_0^t d\tau \left\{ \hat{\rho}\,\partial_\tau \rho + \underbrace{c\,\rho(1-\rho)\left(-e^{-s}e^{\hat{\rho}}+1\right) + (1-c)\,\rho^2\left(-e^{-s}e^{-\hat{\rho}}+1\right)}_{\mathcal{H}} \right\}}$$

One searches for a steady saddle point $\frac{\delta S}{\delta \rho} = \frac{\delta S}{\delta \hat{\rho}} = 0$ ie

$$0 = \frac{\partial \mathcal{H}}{\partial \rho} = -c2\rho\left(1-e^{-s}e^{\hat{\rho}}\right) + 2(1-c)\rho\left(1-e^{-s}e^{-\hat{\rho}}\right)$$

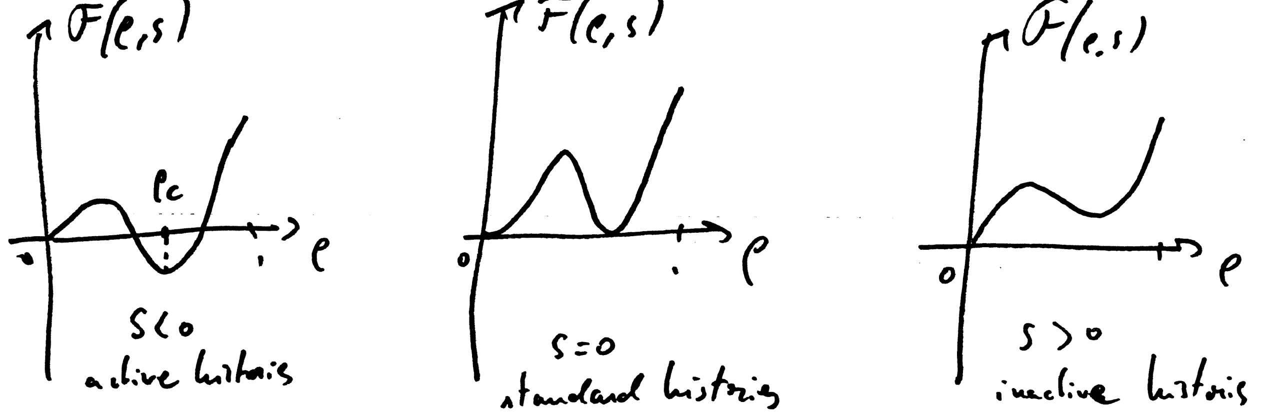
$$0 = \frac{\partial \mathcal{H}}{\partial \hat{\rho}} = -c\rho(1-\rho)e^{-s}e^{\hat{\rho}} + (1-c)\rho^2 e^{-s}e^{-\hat{\rho}} \Rightarrow \boxed{e^{\hat{\rho}} = e^{-s}\sqrt{\frac{1-c}{c}\frac{\rho}{1-\rho}}}$$

Instead of solving the full equations one changes $\min_s S[\hat{\rho},\rho]$ into

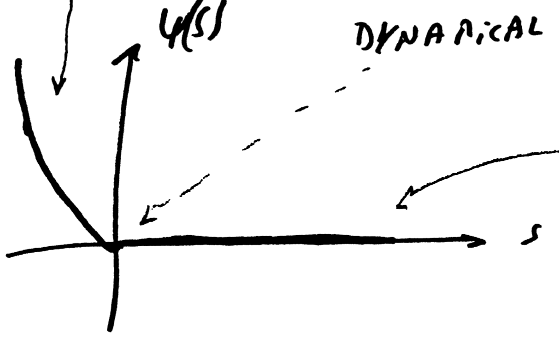
$$\boxed{\psi(s) = -\min_{\rho} \mathcal{F}(\rho,s)} \qquad \text{with } \mathcal{F}(\rho,s) = S_s[\hat{\rho},\rho]$$

$\mathcal{F}(\rho,s)$ is a "dynamical Landau free energy"

$$\boxed{\mathcal{F}(\rho,s) = \dots = c\rho(1-\rho) + (1-c)\rho^2 - 2e^{-s}\sqrt{c(1-c)\rho(1-\rho)}}$$



$s<0$ active histories — $s=0$ standard histories — $s>0$ inactive histories



DYNAMICAL PHASE COEXISTENCE

Competition btw active and inactive histories

Corresponding to the "bubbles" of inactivity in space-time