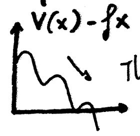
Mini-Project : toy models for the depinning transition

- Problem in continuous space : a particle in a tilted periodic potential

$x \in [0,1]$ with periodic boundary conditions

periodic potential $V(x) = \cos(2\pi p x)$ p integer

constant uniform "drift" force f



The particle slides down

$$\boxed{\partial_t x = -V'(x) + f + \eta(t)}$$

$F(x) = -V'(x) + f$

$\langle \eta(t)\eta(t')\rangle = 2T\,\delta(t-t')$

Corresponding Fokker-Planck equation for the steady state $P_{st}(x)$:

$$0 \;\cancel{\partial_t P(x,t)} = \boxed{-\partial_x\left(F(x) P_{st}(x)\right) + T\partial_x^2 P_{st}(x) = 0}$$

⚠ Note that a naive Boltzmann solution $P_{eq}(x) = e^{-\frac{1}{T}(V(x)-fx)}$ does <u>not</u> work because it does not respect boundary conditions

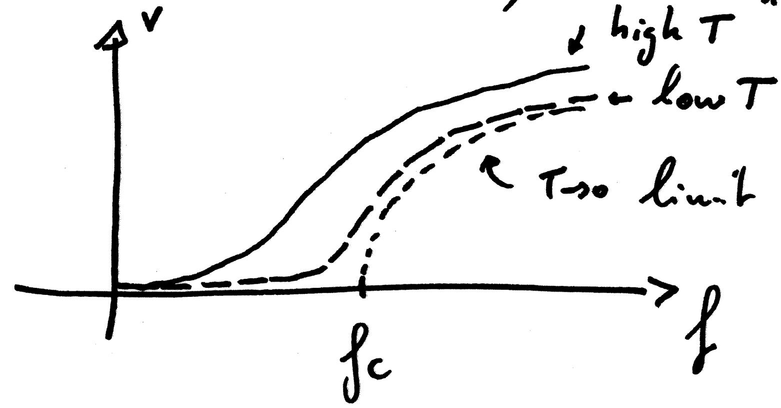
→ The problem is a real 1D <u>non-equilibrium</u> process.

Q 1a : Find the steady-state $P_{st}(x)$

Hint (i) : use that the probability current $F(x)P_{st}(x) - T\partial_x P_{st}$ is a uniform constant j (as seen from the steady state equation)

Hint (ii) : read { P. Le Doussal & V. Vinokur, Physica C (1995) <u>254</u> 63
or S. Scheidl, Z. Phys. B (1995) <u>97</u> 345

Q 1b : Determine the mean velocity $\bar{v} = \langle \partial_t x\rangle_{st}$ as a function of f & T



The limit describes a 1D non-equilibrium $\boxed{\text{phase transition}}$

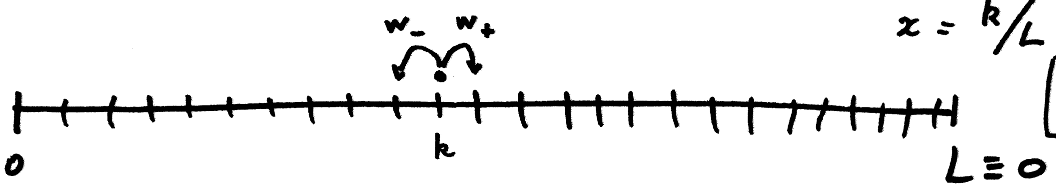
2 - Numerical simulations in discrete time & space :

Time step δt

Lattice with a large number of sites $0 \leq k \leq L$. k: index of the site

$x = k/L \in [0,1]$

$\boxed{\delta x = 1/L}$



Periodic boundary conditions: $k=L$ identified to $k=0$.

Measure the mean-velocity through the total current $\mathring{Q}$

$\mathring{Q} = (\#\text{jumps to the right}) - (\#\text{jumps to the left}) \rightarrow \begin{cases} Q \mapsto Q+1 & \text{for } \curvearrowright \\ Q \mapsto Q-1 & \text{for } \curvearrowleft \end{cases}$

$\rightarrow$ $\bar{v}$ is recovered from $\bar{v} = \frac{1}{t} \langle \mathring{Q} \rangle \delta x$ (average over many runs) and take large t

(The difference btw k and $\mathring{Q}$ is that k is computed modulo L, while $\mathring{Q}$ is not)

- Transition (not obvious to find, since we are out of equilibrium) take

$$\boxed{\begin{aligned} \text{prob}(k \to k+1) &\equiv p_+(k) = \frac{\delta t}{\delta x^2} T e^{-\frac{1}{2T}\left(V\left(\frac{k+1}{L}\right) - V\left(\frac{k}{L}\right) - \frac{f}{L}\right)} \\ \text{prob}(k \to k-1) &\equiv p_-(k) = \frac{\delta t}{\delta x^2} T e^{-\frac{1}{2T}\left(V\left(\frac{k-1}{L}\right) - V\left(\frac{k}{L}\right) + \frac{f}{L}\right)} \end{aligned}}$$

And the probability to stay in k is $1 - p_+(k) - p_-(k)$

$\Rightarrow$ always chose δt so that this probability is > 0 ⚠

This choice is justified by the fact that the master equation for $\mathring{P}(k,t)$

$$\mathring{P}(k, t+\delta t) = p_+(k-1)\mathring{P}(k-1,t) + p_-(k+1)\mathring{P}(k+1,t) + (1 - p_+(k) - p_-(k))\mathring{P}(k,t) -$$

gives, in the large L limit, for $P(x,t) = \frac{1}{\delta x}\mathring{P}\left(\frac{k}{L}, t\right)$, with $x = \frac{k}{L}$, the FP eq. out

$$\partial_t P(x,t) = \partial_x \left((V'(x) - f) P(x,t) \right) + T \partial_x^2 P(x,t)$$

[CHECK THIS] EXPLICITLY and present the derivation in your results.

- Q2a - simulate the particle on a large periodic lattice with the rates above for a potential $V(x) = \cos(2\pi x p)$ as in part 1

Q2b - measure the mean velocity $\bar{v}(f)$ for different temperatures observing the crossover from low to high velocity

Q2c - compare different curves of $\bar{v}(f)$ ——— to the analytical prediction

Mini-Project

3 - Other choice of simulation: continuous space, discrete time

- We have seen in the lecture that the Langevin equation

$$\partial_t x = -V'(x) + f + \eta(t) \qquad \text{with } \langle \eta(t)\eta(t')\rangle = 2T\,\delta(t-t')$$

implies, in discrete time,

$$\boxed{x_{t+\delta t} = x_t - V'(x_t)\,\delta t + f\,\delta t + \mathring{\eta}_t} \qquad \langle \mathring{\eta}_t \mathring{\eta}_{t'} \rangle = 2T \underset{\text{time step}}{\delta t}\, \underset{\text{Kronecker delta}}{\delta_{tt'}}$$

with $\mathring{\eta}_t$ a noise of Gaussian distribution

$$P(\mathring{\eta}) = \frac{1}{\sqrt{4\pi T \delta t}}\, e^{-\frac{1}{2}\frac{\mathring{\eta}^2}{2T\delta t}} \qquad (*)$$

- You can thus simulate the equation in discrete time and continuous space $x_t \in [0,1]$ (using periodic boundary conditions) by drawing at each time step a value of the noise $\mathring{\eta}_t$ distributed with $(*)$.

- To measure the velocity, keep track of Q_t which is the same as x_t excepted that it is not taken modulo 1.

You can then evaluate as: $\bar{v} = \frac{1}{t}\langle Q_t \rangle$

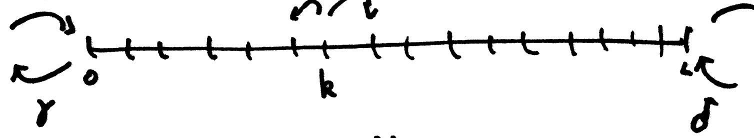
in the large time limit — average over many runs of the simulation

→ Answer the same questions as in part 2.

4 - Other situation (Bonus): open system $F(x) = -V'(x)$

$$\partial_t P = \partial_x\big(V'(x)P(x,t)\big) + T\,\partial_x^2 P$$

system driven by boundaries. Corresponds to the microscopic evolution



$$p_\pm(k) = \frac{\delta t}{\delta x^2}\, T\, e^{-\frac{1}{2T}\left(V\left(\frac{k\pm 1}{L}\right) - V(k)\right)}$$

You can try to answer the same questions as in the periodic case.

Ref: See part 1.1 of J. Tailleur, J. Kurchan, VL J. Phys. A 41 505001 (2008)