Exercice 1 - correction

Homework 1 - Elements of correction

1. Particle in a tilted potential

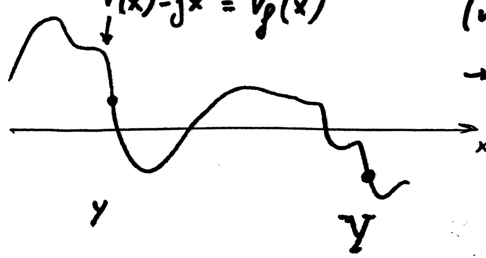
$x \in [0,1]$ periodic b.c.

$\partial_t x = -V'(x) + f + \eta(t) \qquad F(x) = -V'(x) + f \qquad \langle \eta(t)\eta(t')\rangle = 2T\,\delta(t-t')$

st.st. : $-\partial_x\big(F(x)P_{st}(x)\big) + T\,\partial_x^2 P_{st}(x) = 0$

Instead of solving $P_{st}(x)$ as in Le Doussal & Vinokur or as in Scheidl, one may follow a 1st passage time approach (mean ___ : MFPT)

$V(x) - fx \equiv V_f(x)$



(note that $V_f(x)$ is not a continuous periodic function)
→ one transforms the problem into a problem with $x \in \mathbb{R}$

| y = starting point
| Y = passage point

$t_1(y \to Y)$ = mean 1st passage time in Y, starting from y at time 0.

One has $\underbrace{\langle t_1(y \to Y)\rangle}_{\tau_1(y)} = \int_0^{+\infty} dt\; t\; \mathrm{Prob}\big(t_1(y\to Y) = t\big)$

$= \int_0^{+\infty} dt\; t \cdot \Big(-\partial_t \mathrm{Prob}\big(t_1(y\to Y) \geq t\big)\Big) = 0 + \int_0^{+\infty} dt\; \mathrm{Prob}\big(t_1(y\to Y)\geq t\big)$ → This is a cumulative distribut° funct°

$\underbrace{\mathrm{Prob}\big(t_1(y\to Y)\geq t\big)}_{\equiv Q(y,t)} = \int_{-\infty}^{Y} dy'\; \underbrace{P(y',t|y,0)}$

condit. prob. density that the particle is in y' at time t, knowing that it was in y at time 0.

= probability that the particle goes in Y for the 1st time after time t, having started in y at $t=0$.

= probability that the particle is still in any $y' \in]-\infty, Y[$ at time t, having started at time 0 in y.

Note: with the notations above: $\boxed{\tau_1(y) = \int_0^{+\infty} dt\; Q(y,t)}$

It verifies the Backwards Fokker-Planck equation with respect to the initial position y

$$\boxed{\partial_t P(y',t|y,0) = F(y)\,\partial_y P(y',t|y,0) + T\,\partial_y^2 P(y',t|y,0)}$$ (see lecture p. 1.20)

$\int_{-\infty}^{Y} dy'$

Hence $\partial_t Q(y,t) = F(y)\,\partial_y Q(y,t) + T\,\partial_y^2 Q(y,t)$ with $Q(Y,t) = 0 \;\; \forall t$ [because $t_1(Y\to Y) = 0$]

$\int_0^{+\infty} dt$: $\underbrace{Q(y,+\infty)}_{=0} - \underbrace{Q(y,0)}_{=1} = F(y)\,\partial_y \tau_1(y) + T\,\partial_y^2 \tau_1(y)$

Finally, the equation on the MFPT is: $\boxed{F(y)\,\partial_y \tau_1(y) + T\,\partial_y^2 \tau_1(y) = -1, \quad \tau_1(Y) = 0}$

(Pontryagin equation)

One finds check this!! → $\boxed{\tau_1(y) = \frac{1}{T}\int_y^{Y} dy' \int_{-\infty}^{y'} dy''\, \frac{Z^f(y'')}{Z^f(y')}}$

$Z^f(y_0) = e^{-\frac{1}{T}V_f(y_0)} = e^{-\frac{1}{T}(V(y_0) - f y_0)}$

= Boltzmann weight if one were in equilibrium

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1- Use of the formula for the MFPT : $p=1$ to simplify

one could of course take any period p.

in our case $$\frac{Z\delta(y'')}{Z\delta(y')} = \underbrace{e^{\beta f(y''-y')}}_{\text{this part is not periodic}} \cdot \underbrace{e^{-\beta[V(y'')-V(y')]}}_{\text{this part is periodic of period 1}}$$

in the expression of $\tau_1(y)$ we thus set $y''=y'-y_0$, $y_0\in[0,+\infty[$ so that, inverting integrations

$$\tau_1(0) = \beta\int_0^{+\infty} dy_0\, e^{-\beta f y_0} \underbrace{\int_0^{Y} dy'\, \underbrace{e^{-\beta[V(y'-y_0)-V(y')]}}_{\text{periodic: invariant by translation } y'\to y'+1}}_{\text{for } Y\in\mathbb{N} \text{ this integral is thus } Y\times\int_0^1 dy'\, e^{-\beta[V(y'-y_0)-V(y')]}}$$

We thus obtain that for $Y\in\mathbb{N}$, the MFPT from 0 to Y is proportional to Y.
As Y represents the distance from 0 to Y, this allows to identify the mean velocity $\bar{v}$ as $\bar{v} = \frac{Y}{\tau_1(0\to Y)} = \frac{1}{\tau_1(0\to 1)}$. In other words,

$$\boxed{\bar{v}^{-1} = \beta\int_0^{+\infty} dy_0\, e^{-\beta f y_0}\int_0^1 dy'\, e^{-\beta[V(y'-y_0)-V(y')]}}$$

This expression is generic for any 1-periodic potential $V(x)$.
Computations can be pushed forward for the cosine potential:

$$V(y'-y_0) - V(y') = 2\sin(\pi y_0)\sin[\pi(y_0-2y')]$$

Using now that for any periodic function $\varphi(y)$ of period 1, $\int_0^1 dy'\varphi(y'+a) = \int_a^{1+a} ds'\varphi(s') = \int_0^1 dy'\varphi(y')$
one can translate $y'\to y'-y_0/2$ in the integral and this yields

$$\bar{v}^{-1} = \beta\int_0^{+\infty} dy_0\, e^{-\beta f y_0}\underbrace{\int_0^1 dy'\, e^{2\beta\sin(\pi y_0)\sin(2\pi y')}}_{I_0(2\beta\sin\pi y_0)}$$

$I_0(x)$ = Bessel function I

$$\boxed{\bar{v}^{-1} = \beta\int_0^{+\infty} dy_0\, e^{-\beta f y_0}\, I_0(2\beta\sin\pi y_0)}$$

See the mathematica file for some plots