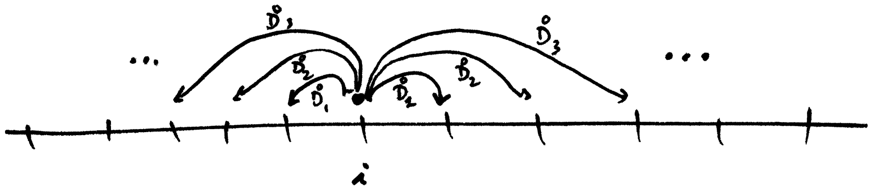
Partial Exam

Exercice 1 - examples of 1D diffusion

Consider a particle, of position $i \in \mathbb{Z}$, jumping symmetrically



to site $\begin{cases} i+k \quad (k \geq 1) & \text{with probability} \quad \mathring{D}_k \\ i-k \quad (k \geq 1) & \quad \mathring{D}_k \end{cases}$

between t and $t+dt$.

1-1- a What is the probability to stay in i between t and $t+dt$?

b What condition do the $\mathring{D}_k$'s have to verify for the process to be well-defined?

1-2- a We introduce $x = i\,\delta x$ where $\delta x > 0$ is a lattice spacing. Write the equation of evolution for the probability $\mathring{P}(x,t)$ for the particle to be at position x at time t.

b Take the continuum limit $\delta x \to 0$, $\delta t \to 0$:
What is the diffusion constant D of the process?
How δx and δt have to scale ($\{\mathring{D}_k\}$ being fixed), to have a diffusion?

1-3- a Example: $\mathring{D}_k = \dfrac{\ell}{\delta x} \exp\left(-\dfrac{k\,\delta x}{\ell}\right)$

. What is the interpretation of ℓ?

b . At fixed δx, what does ℓ has to verify? [Rq: $\sum_{k=1}^{+\infty} a^k = \frac{a}{1-a}$ for $a<1$]

c . Determine D.

Exercice 2 - Brownian Motion and Scalings

Consider a Brownian motion $B_t \;\Big|_{B_0=0}^{t\geq 0}$ (time and space are continuous)

with $\langle (B_t - B_{t'})^2 \rangle = |t'-t|$

2-1 We write the probability density for trajectories as

$$\mathbb{P}[(B_\tau)_{0\leq\tau\leq t_f}] \propto \exp(-S[B]) \quad \text{. What is } S[B] \text{ ?}$$

[give its expression]

t_f: final time

2-2 Consider now another Brownian motion $V_t \;\Big|_{V_0=0}^{t\geq 0}$ with

with $\langle (V_t - V_{t'})^2 \rangle = D\,|t'-t|$

a. We define the rescaled process

$$\tilde{V}_t = V_{at} \qquad (a>0)$$

Find the 'action' $S[\tilde{V}]$ for $\mathbb{P}[(\tilde{V}_t)_{0\leq\tau\leq t_f}]$.

b. Find a constants c_1 so that the actions for $\tilde{V}_t$ and for $c_1 B_t$ are the same

In this case, we write that

$$\boxed{V_{at} \overset{d}{=} c_1 B_t} \quad (*) \quad \text{(an "equality in distribution")}$$

2-3 In the same way, consider two white noises η_t and ξ_t with $\langle \eta_t \eta_{t'} \rangle = \delta(t'-t) \qquad \langle \xi_t \xi_{t'} \rangle = 2D\,\delta(t'-t)$

Find a constant c_2 such that

$$\boxed{\xi_{at} \overset{d}{=} c_2\, \eta_t} \quad (**) \qquad (a>0)$$

2-4 - Consistency check: verify that your results for c_1 and c_2 are correct by computing

$$\langle (V_{at} - V_{at'})^2 \rangle \quad \text{and} \quad \langle \xi_{at}\, \xi_{at'} \rangle$$

> Remark: you might need the identity $\delta(at) = \frac{1}{a}\delta(t)$
> Bonus: prove it.

a. directly

b. using the equalities in distribution $(*)$ and $(**)$ above.

2.5 [Difficult] A particle in a parabola + a brownian motion

Consider a particle of position y in a potential

$$\mathcal{H}_V(y,\ell) = c\,\frac{y^2}{2\ell} + V(y) \quad \text{where } V(y) \text{ is Brownian,}$$

with $\langle (V(y)-V(y'))^2\rangle = D|y'-y|$

It can be shown that it represents the extremity of a polymer of lenght ℓ in a random environment, of distribution

$$P_V(y,\ell) \propto e^{-\frac{1}{T}\mathcal{H}_V(y,\ell)} \qquad (T \text{ is the temperature})$$

a. What does represent

$$B(\ell; cDT) = \left\langle \frac{\int dy\; y^2\, e^{-\frac{1}{T}\mathcal{H}_V(y,\ell)}}{\int dy\; e^{-\frac{1}{T}\mathcal{H}_V(y,\ell)}} \right\rangle_{(*)} \quad ?$$

averages over the realizations of V

b. Perform a change of variables $y = a\bar{y}$ such that

$$\mathcal{H}_V(a\bar{y},\ell) \overset{d}{=} \mathcal{E}_1 \times \left[\frac{\bar{y}^2}{2} + B(\bar{y})\right] \qquad \langle (B(\bar{y})-B(\bar{y}'))^2\rangle = |\bar{y}'-\bar{y}|$$

(You have to find $\mathcal{E}_1$ and a as a function of c, D, T, ℓ)

c. Using this change of variable in (*) above, find the large ℓ asymptotics for $B(\ell; cDT)$.

Hint: you'll find $B(\ell) \sim \ell^{4/3}$ linked to the KPZ exponent $2/3$.

d. Interpret the high and low temperature behaviors

Exercice 3 - Central Limit Theorem

Consider a random variable x of variance σ and average μ:

$$\begin{cases} \langle x \rangle = \mu \\ \langle (x - \langle x \rangle)^2 \rangle = \sigma \end{cases}$$

We note $P(x)$ the distribution of x.

We consider the statistics of $X_N = \dfrac{\overbrace{x_1 + \dots + x_N}^{\text{independent realizations of } x}}{N}$

You may think of X_N as a position at time N, and to $x_1 \dots x_N$ as increments.

3_1. Cumulant-generating functions

$$\Psi_x(s) = \log \langle e^{-sx} \rangle$$

a_ **Examples**:

Show that $P(x) = \delta(x-\mu) \Rightarrow \Psi_x(s) = -s\mu$

$$P(x) \propto \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) \Rightarrow \Psi_x(s) = -s\mu + \frac{1}{2}\sigma^2 s^2$$

b_ **Property of linearity**: Show that $\Psi_{\lambda_1 x_1 + \lambda_2 x_2}(s) = \Psi_{x_1}(\lambda_1 s) + \Psi_{x_2}(\lambda_2 s)$

if x_1 and x_2 are independent

3_2. Limit Central Theorem: (C.L.T)

a_ If $\Psi_x(s) = -\mu s + \frac{1}{2}\sigma^2 s^2 + K_3 s^3 + \dots$

determine $\Psi_{X_N}(s) = \overset{?}{\dots} s + \overset{?}{\dots} s^2 + \overset{?}{\dots} s^3 + \dots$ Comment on the limit $N \to \infty$.

b_ Read in this expression the mean and the variance of X_N

c_ Define $Y_N = (X_N - \mu) N^{1/2}$

Find $\Psi_{Y_N}(s)$ in a form similar to the previous expressions.

Take the $N \to \infty$ limit: what is $\lim\limits_{N \to \infty} \Psi_{Y_N}(s)$?

What does it mean for the distribution of Y_N at large N?

d_ Explain how one can write

$$X_N \overset{d}{=} \mu + N^{-1/2} b$$

with b a random variable of normal gaussian distribution

This is the C.L.T.

3.3 [difficult]

Case where x does not have a finite variance:

$$P(x) = \frac{\alpha}{x^{\alpha+1}} \; , \quad 1 < \alpha < 2 \quad , \; x \in [1, +\infty[\; \equiv I$$

a- Check that $P(x)$ is normalized, and determine $\mu = \langle x \rangle = \int_I dx \; x \, P(x)$

b- Try to compute $\langle x^2 \rangle$; what happens?

c- (very difficult) Show that, in the small s limit

$$\Psi_x(s) = -s\mu + C_\alpha s^\alpha + \quad s^2 \quad + \dots$$

d- In the same way as previously, show that

$$X_N \stackrel{d}{=} \mu + N^{-\gamma} y \qquad \text{[you have to find } \gamma \text{]}$$

where, in the large N limit, the distribution of y is independent of N.

This is an extension of the C.L.T. for variables which do not have a finite variance.

Exercice 4 - Statistics of the activity

We consider a Markov evolution on configurations $\{\mathcal{C}\}$.
We are interested in the activity K, on a time window $[0,t]$
(= # events between 0 and t).

4.1 - a. Explain in details why

$$\boxed{\partial_t P(\mathcal{C},K,t) = \sum_{\mathcal{C}'} W(\mathcal{C}'\to\mathcal{C})\, P(\mathcal{C}',K-1,t) - \underbrace{r(\mathcal{C})}_{\sum_{\mathcal{C}''} W(\mathcal{C}\to\mathcal{C}'')} P(\mathcal{C},K,t)}$$

b. From this equation of evolution, determine

$$\partial_t \langle K\rangle = \partial_t \sum_{\mathcal{C},K} K P(\mathcal{C},K,t)$$

c. Find an observable $\mathcal{O}(\mathcal{C})$ such that $\lim_{t\to\infty} \frac{1}{t}\langle K\rangle = \overbrace{\langle \mathcal{O}(\mathcal{C})\rangle}^{\text{average in the steady state}}$

4.2. Introduce $\hat{P}(\mathcal{C},s,t) = \sum_K e^{-sK} P(\mathcal{C},K,t)$

a. Find an equation of evolution for $\hat{P}(\mathcal{C},s,t)$. Does it preserve probability?

b. Show that it takes the form

$$\partial_t |\hat{P}(t)\rangle = \mathbb{W}_s |\hat{P}(t)\rangle \quad \text{with}$$

$$(\mathbb{W}_s)_{\mathcal{C}\mathcal{C}'} = e^{-s} W(\mathcal{C}'\to\mathcal{C}) - r(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}$$

c. Explain in details why

$$\langle e^{-sK}\rangle = \sum_{\mathcal{C}} \hat{P}(\mathcal{C},s,t)$$

d. The cumulant generating function is $\psi(s) = \lim_{t\to\infty} \frac{1}{t}\log\langle e^{-sK}\rangle$

Find an expression of $\psi(s)$ in relation to the spectrum of the matrix $\mathbb{W}(s)$.

4.3. An example: $\{\mathcal{C}\} = \{0,1\}$ $\qquad 0 \underset{1-c}{\overset{c}{\rightleftarrows}} 1$

a. Find $\mathbb{W}_s$ (it is a 2x2 matrix)

b. Determine $\psi(s)$. [use that the eigenvalues of $\mathbb{W}$ are the roots of $\mathbb{W} - X\mathbb{1}$].

c. Compute and interpret $\psi(0)$, $\psi'(0)$, $\psi''(0)$.

d. With $|P(t=0)\rangle = \binom{1}{0}$, find $|\hat{P}(s,t)\rangle$.