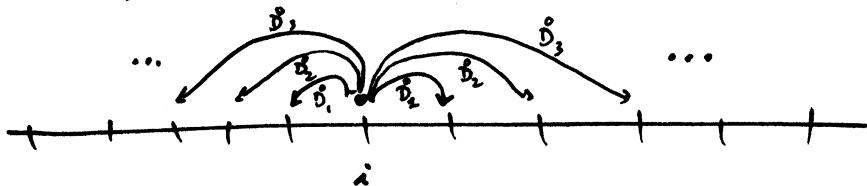


Exercise 1 - examples of 1D diffusion

Consider a particle, of position  $i \in \mathbb{Z}$ , jumping asymmetrically



to site  $\begin{cases} i+k & (k \geq 1) \\ i-k & (k \geq 1) \end{cases}$  with probability  $\begin{cases} \hat{D}_k & \\ \hat{D}_{-k} & \end{cases}$

between  $t$  and  $t+dt$ .

1-1. a What is the probability to stay in  $i$  between  $t$  and  $t+dt$ ?

b What condition do the  $\hat{D}_k$ 's have to verify for the process to be well-defined?

1-2. a We introduce  $x = i\delta x$  where  $\delta x > 0$  is a lattice spacing.  
Write the equation of evolution for the probability  $\hat{P}(x, t)$  for the particle to be at position  $x$  at time  $t$ .

b Take the continuum limit  $\delta x \rightarrow 0$ ,  $\delta t \rightarrow 0$ :

What is the diffusion constant  $D$  of the process?

How  $\delta x$  and  $\delta t$  have to scale ( $\{\hat{D}_k\}$  being fixed), to have a diffusion?

1-3. a Example :  $\hat{D}_k = \frac{l}{\delta x} \exp\left(-\frac{k\delta x}{l}\right)$

. What is the interpretation of  $l$ ?

b . At fixed  $\delta x$ , what does  $l$  has to verify? [Rq:  $\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$  for  $a < 1$ ]

c . Determine  $D$ .

Exercise 2: Backwards Fokker-Planck equation

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- . Consider a Langevin equation  $\partial_t x = F(x) + \gamma(t)$   
with  $\langle \gamma(t) \gamma(t') \rangle = 2D \delta(t-t')$

and the probability density,

$$P(x, t | x_0, t_0)$$

of being in  $x$  at time  $t$ , having started from  $x_0$  at time  $t_0$ .

- . Following the same route as the lecture, p 1.10, obtain an evolution equation  $\partial_{t_0} P(x, t | x_0, t_0) = \dots \partial_{x_0} \dots \partial_{x_0}^2 \dots P$

### Exercise 3 - Central Limit Theorem

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Consider a random variable  $x$  of variance  $\sigma^2$  and average  $\mu$ :

$$\begin{cases} \langle x \rangle = \mu \\ \langle (x - \langle x \rangle)^2 \rangle = \sigma^2 \end{cases}$$

We note  $P(x)$  the distribution of  $x$ .  
independent realizations of  $x$

We consider the statistics of  $X_N = \frac{x_1 + \dots + x_N}{N}$

You may think of  $X_N$  as a position at time  $N$ , and to  $x_1, \dots, x_N$  as increments.

#### 3-1. Cumulant-generating-functions

$$\Psi_x(s) = \log \langle e^{-sx} \rangle$$

a- Examples:

$$\text{Show that } P(x) = \delta(x - \mu) \Rightarrow \Psi_x(s) = -s\mu$$

$$P(x) \propto \exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}\right) \Rightarrow \Psi_x(s) = -s\mu + \frac{1}{2} \sigma^2 s^2$$

b- Property of linearity: Show that  $\Psi_{\lambda_1 x_1 + \lambda_2 x_2}(s) = \Psi_{x_1}(\lambda_1 s) + \Psi_{x_2}(\lambda_2 s)$   
if  $x_1$  and  $x_2$  are independent

#### 3-2. Limit Central Theorem : (C.L.T.)

a- If  $\Psi_x(s) = -\mu s + \frac{1}{2} \sigma^2 s^2 + K_3 s^3 + \dots$

determine  $\Psi_{X_N}(s) = \dots s + \dots s^2 + \dots s^3 + \dots$  Comment on the limit  $N \rightarrow \infty$ .

b- Read in this expression the mean and the variance of  $X_N$

c- Define  $Y_N = (X_N - \mu) N^{1/2}$

Find  $\Psi_{Y_N}(s)$  in a form similar to the previous expressions.

Take the  $N \rightarrow \infty$  limit: what is  $\lim_{N \rightarrow \infty} \Psi_{Y_N}(s)$ ?

What does it mean for the distribution of  $Y_N$  at large  $N$ ?

d- Explain how one can write

$$X_N \stackrel{d}{=} \mu + N^{1/2} b \quad \text{with } b \text{ a random variable of normal Gaussian distribution}$$

This is the C.L.T. .