Large deviations in 1d exclusion processes: Mapping non-equilibrium onto equilibrium

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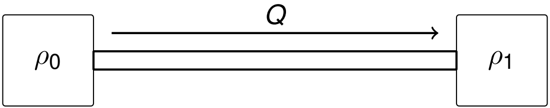
Outline

1. Motivations
2. Exclusion Processes
   - Operator approach
   - Large deviation functions
3. Mapping non-equilibrium to equilibrium
   - For the integrated current
   - For the density profile

Motivations



$$\text{Prob}(\mathcal{C}) \propto \exp\left\{-\frac{\text{energy}(\mathcal{C})}{\text{temperature}}\right\}$$ cannot describe

- Non-equilibrium steady-state

$$\text{Prob}[\rho(x)]$$

- Equilibrium fluctuations of dynamical observables

$$\text{Prob}[Q]$$

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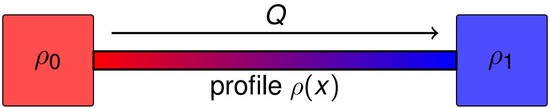

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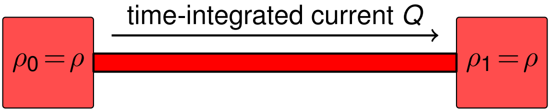

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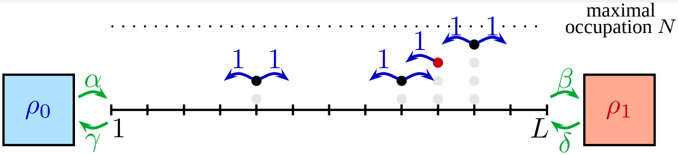
Exclusion processes



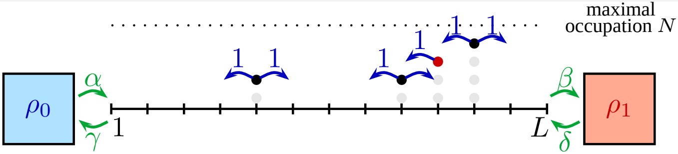
- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution

$$\partial_t P(\{n_i\}) = \sum_{n_i'} \left[W(n_i' \to n_i) P(\{n_i'\}) - W(n_i \to n_i') P(\{n_i\}) \right]$$

- Large deviation function of the time-integrated current Q

$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \qquad (\Leftrightarrow \text{determining } P(Q))$$

Exclusion processes



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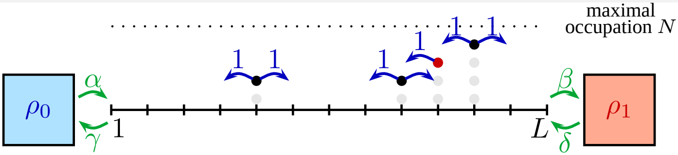

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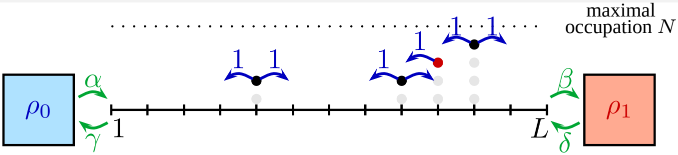
Operator representation

[Schütz & Sandow PRE **49** 2726]

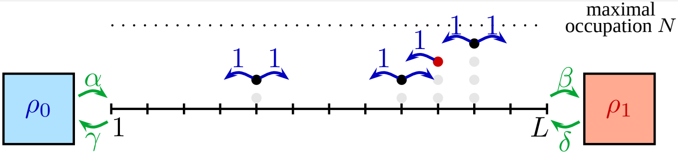


$$\partial_t P \;=\; \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\le k\le L-1} \Big[S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k)\Big]$$
$$+\,\alpha\,\big[S_1^+ - (1-\hat{n}_1)\big] + \gamma\,\big[S_1^- - \hat{n}_1\big]$$
$$+\,\delta\,\big[S_L^+ - (1-\hat{n}_L)\big] + \beta\,\big[S_L^- - \hat{n}_L\big]$$

$S^\pm$ and $S^z = \hat{n} - \frac{N}{2}$ are **spin operators** (with $j = \frac{N}{2}$)

Operator representation



$$\left\langle e^{-sQ} \right\rangle \sim e^{t\psi(s)} \qquad \text{with} \qquad \psi(\lambda) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

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Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

For **periodic boundary conditions**

- large deviation function

$$\psi(s) = \underbrace{L^{-1}\rho(1-\rho)s^2}_{\text{order } 0} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} \quad \text{with} \quad u = -\tfrac{1}{2}\rho(1-\rho)s^2$$

- universal function

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

- scaling of cumulants of the total current Q_{tot}

$$\tfrac{1}{t}\langle Q_{\text{tot}}^2\rangle \sim L$$

$$\tfrac{1}{t}\langle Q_{\text{tot}}^{2k}\rangle \sim L^{2k-2} \qquad (k \geq 2)$$

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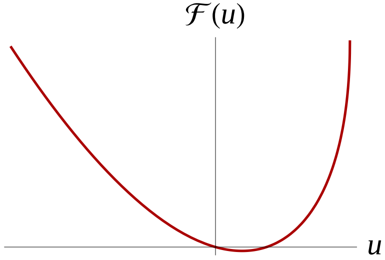

With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
Driving field E

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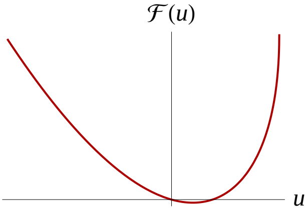
Periodic boundary conditions
Driving field E

For the WASEP:
$D = 1$, $\sigma = 2\rho(1-\rho)$

Large deviation function

$$\psi(s) = \underbrace{\tfrac{1}{2}s(s-E)\frac{\langle Q^2\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\substack{\text{small fluctuations}\\ \text{(determinant)}}} \quad \text{with} \quad u = \underbrace{-s(s-E)\frac{\sigma\sigma''}{16D^2}}_{\text{can become } > 0}$$



Dynamical phase transition
between
stationnary and non-stationnary
profiles

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Microscopic approach [Imparato, VL, van Wijland, arXiv:0911.0564]

Large deviations of the current

$$\psi(\lambda) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

$$\mathbb{W}(s) = \overbrace{\sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}} + \alpha\left[S_1^+ - (1-\hat{n}_1)\right] + \gamma\left[S_1^- - \hat{n}_1\right] + \delta\left[S_L^+ e^{s} - (1-\hat{n}_L)\right] + \beta\left[S_L^- e^{-s} - \hat{n}_L\right]$$

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Local transformation

$$\mathcal{Q}^{-1}\mathbb{W}(s)\mathcal{Q} = \sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1} + \alpha'\left[S_1^+ - (1-\hat{n}_1)\right] + \gamma'\left[S_1^- - \hat{n}_1\right] \underbrace{+\delta'\left[S_L^+ e^{s'} - (1-\hat{n}_L)\right] + \beta'\left[S_L^- e^{-s'} - \hat{n}_L\right]}_{\text{describes contact with reservoirs of same densities}}$$

Result

[Imparato, VL, van Wijland, **PRE** 80 011131]

Large deviation function

$$\psi(s) = \underbrace{\frac{1}{L}\mu(s)}_{\text{saddle}} + \overbrace{\underbrace{\frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(s)\right)}_{\text{fluctuations}}}^{\textbf{same } \mathcal{F} \textbf{ as at eq.}}$$

$$\mu(\lambda) = -\big(\text{arcsinh}\,\sqrt{\omega}\big)^2$$

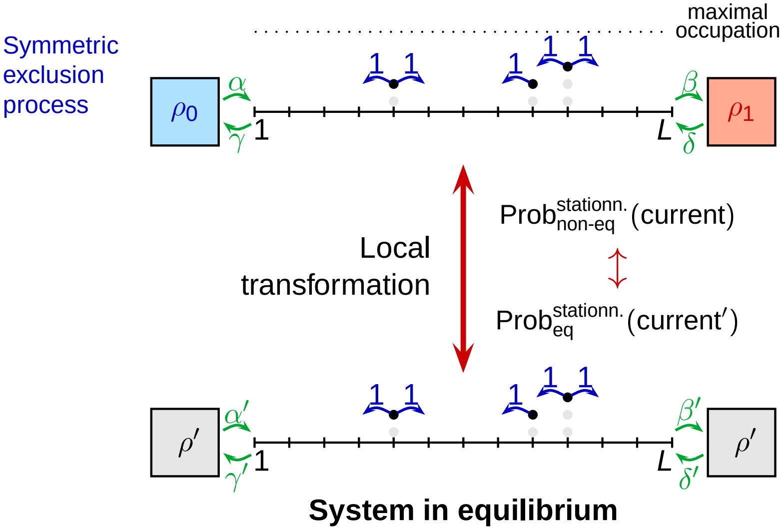
$$\omega = (1 - e^{\lambda})\big(e^{-\lambda}\rho_0 - \rho_1 - (e^{-\lambda} - 1)\rho_0\rho_1\big)$$

Physical remark

In contact with reservoirs, neither SSEP nor KMP present a 'dynamical phase transition' ($\Leftarrow$ no singularity in $\psi(s)$).

For the current

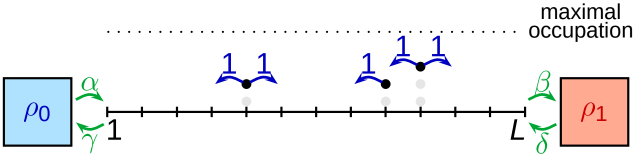
[Imparato, VL, van Wijland, **PRE** 80 011131]



For the density profile

[Tailleur, Kurchan, VL, JPA **41** 505001]

Symmetric exclusion process

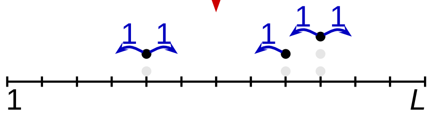


Non-local transformation

$\mathrm{Prob}^{\text{stationn.}}_{\text{non-eq}}(\text{profile})$

$\updownarrow$

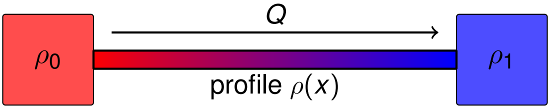
$\mathrm{Prob}^{\text{stationn.}}_{\text{eq}}(\text{profile}')$



System in equilibrium

Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]

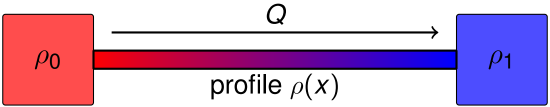


Boundary-driven transport model:

- long-range correlations
- breaking of time-reversal symmetry

Non-local mapping

[Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

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Non-local mapping to equilibrium:

- accounts for long-range correlations
- (density gradient)$_{\text{non-eq.}} \longleftrightarrow$ (fixed density)$_{\text{eq.}}$
- yields Prob[$\rho(x)$] through a extremalization principle

Summary

Microscopic approach:

- operator formalism
- large deviation function

Macroscopic approach:

- action of fluctuating hydrodynamics
- saddle-point method, instantons
- integration of fluctuations (dynamical phase transition)

$\rightarrow$ Eq$\leftrightarrow$non-eq mapping in higher dimensions?
$\rightarrow$ More generic systems of interacting particle?
$\rightarrow$ Crossover to KPZ? Universal fluctuations?