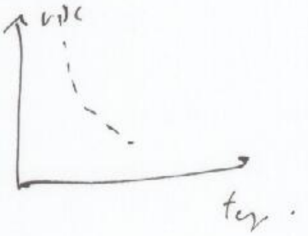
1 - Motivations -

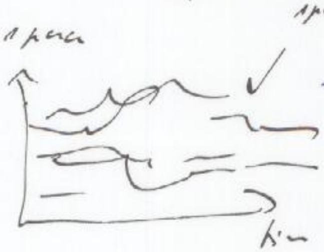
1 - 1 Glassy Phenomena :

- Glasses :



- Dynamical perspective :

- Nat of the states ; i.e $\neq$ critical slow-down
  in particular: no evidence for a static growl
  space-time bubbles of static length
  inactive



1 d system
1 d time

2 - Approach: statistics over histories - Ruelle & com. for

2.1 . Stochastic dynamics : (no quantum aspects)

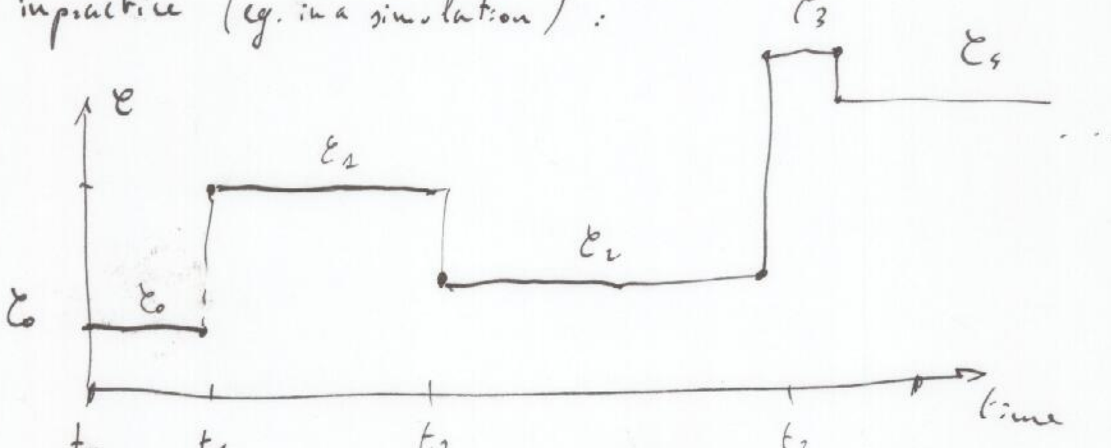
configuration $\mathcal{C}$, jump rates $W(\mathcal{C} \to \mathcal{C}')$

for instance : $\mathcal{C} = (n_1 \dots n_L)$, $n_k = 0$ or 1 Lattice Gas $\cancel{W(n \to n_k \pm 1) = c\,(n_{k-1} + n_{k+1}) , (1-c)(n_{k-1}+n_{k+1})}$

evolution in time of the probability of being in $\mathcal{C}$ at time t:

$$\partial_t P(\mathcal{C},t) = \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C})\, P(\mathcal{C}',t) - \boxed{\underbrace{\sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}')}_{\equiv\, r(\mathcal{C})}}\, P(\mathcal{C},t)$$

in practice (eg. in a simulation) :



KS ⟷ Julien

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2.2 - Activity :

× How can one characterize the "complexity" of an history
or the "activity"

Simplest approach : Consider $\boxed{K = \#\text{ of events (i.e. jumps "}\mathcal{C}\to\mathcal{C}'\text{")}}$ btw 0 and t.
K is a fluctuating observable (depends on the history)
One expects that $\langle K\rangle$ is $\sim t$ indeed: for large t

Rq: the statistics (p.d.f.) of K / or cumulants of K will in fact play an important role.

× Aim : classify histories according to K. $P(\mathcal{C}, K, t)$

$$\partial_t P(\mathcal{C},K,t) = \sum_{\mathcal{C}'} W(\mathcal{C}'\to\mathcal{C})\, P(\mathcal{C}', K-1, t) - r(\mathcal{C})\, P(\mathcal{C},K,t)$$

$P(\mathcal{C},K,t)$ difficult to tackle:

One do not lose physical insight going microcanonical → ~~…~~canonical ensemble

× (Dynamical) canonical ensemble: Legendre Transform.

$$\hat{P}(\mathcal{C},s,t) = \sum_K e^{-sK} P(\mathcal{C},K,t) \qquad \left(\leftrightarrow\ P(\mathcal{C},K,t) = \int_{i\mathbb{R}^+} ds\, e^{sK}\,\hat{P}(\mathcal{C},s,t)\right)$$

2.3 Effective evolution at s :

$$\boxed{\partial_t \hat{P}(\mathcal{C},s,t) = \sum_{\mathcal{C}'} e^{-s} W(\mathcal{C}'\to\mathcal{C})\, \hat{P}(\mathcal{C}',s,t) - r(\mathcal{C})\, \hat{P}(\mathcal{C},s,t)}$$

what can we obtain from this ?

Non-stochastic evolution
$\hat{P} \sim e^{\psi_K(s)t}$ $(t\to\infty)$, $\psi_K(s) = \max \mathrm{Sp}\, \mathbb{W}_K$
$(\mathbb{W}_K)_{\mathcal{C}\mathcal{C}'} = e^{-s} W(\mathcal{C}'\to\mathcal{C}) - r(\mathcal{C})\,\delta_{\mathcal{C}\mathcal{C}'}$

ε- All cumulants of K :

$$\langle e^{-sK}\rangle = \sum_{\text{histories}} e^{-sK} P(\text{history}) = \sum_{\mathcal{C}}\sum_K e^{-sK} P(\mathcal{C},K,t) = \sum_{\mathcal{C}} \hat{P}(\mathcal{C},s,t) \underset{t\to\infty}{\sim} e^{+t\psi_K(s)}$$

$\psi_K(s)$ is the cumulant generating function of K

$$\boxed{\frac{1}{t}\langle K^k\rangle_c = (-1)^k \frac{\partial^k}{\partial s^k}\psi_K(s)}$$

in note this for $t\to\infty$ (one can use max Sp.)
[illegible] … also true

a- Interpretation:

$$Z_K(s,t) = \sum_{\mathcal{C},K} e^{-sK} P(\mathcal{C},K,t) \quad \Longleftrightarrow \text{ dynamical partition function}$$

$$Z_K(s,t) \sim e^{t\psi_K(s)} \quad \rightarrow \psi_K(s) \text{ dynamical free energy.} \quad (\text{large } t \Leftrightarrow \text{large } L)$$

in particular: if ψ_K is non-analytic: this will indicate a phase transition

b- Dynamical ensembles: $s \Leftrightarrow K$

$$\begin{cases} s>0: \text{ histories with "low activity" (to interpret)} & < \langle K \rangle \\ s=0: \text{ normal steady state}, & \langle K \rangle \\ s<0: \text{ histories with "high activity"} & > \langle K \rangle \end{cases}$$

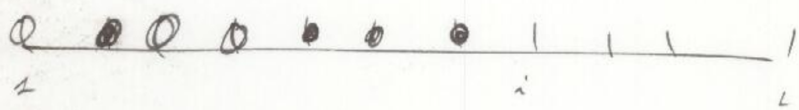
CHECK!

3 - Analytical Results -

3.1 Kinetically Constrained Models - (Fredrickson-Andersen)

inactive active

$n_i = 0$ or 1



L sites

$$W(n_i \to n_i + 1) = c(n_{i+1} + n_{i-1})/2 \qquad = c$$

$$W(n_i \to n_i - 1) = \underbrace{(1-c)(n_{i+1}+n_{i-1})/2}_{\text{constrained}} \qquad = \underbrace{1-c}_{\text{unconstrained}}$$

~~activity~~ evolution possible only if at least one neighbor is active.

- Presents typical features of glassy phenomena:

  stretched exponential for relaxation of correlation function

  Hysteresis in temperature cycles, ageing

- However: the steady, equilibrium state is the same for constrained & unconstrained models!] trivial to check

  (Namely: product of Poissonians of density c in each site).

  Hence: | this is an explicit case where the knowledge of the equilibrium state (or landscape) gives no~~thing~~ information about glassy features!

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3.2 Mean field approach:

All sits are equivalents. n = total occupation, $1 \le n \le L$

$$\begin{cases} W(n \to n+1) = c\,(L-n)\; n/L \\ W(n \to n-1) = (1-c)\; n\; n/L \end{cases}$$

$\uparrow$ counting factor $\uparrow$ facilitation

$r(n) = c\,(L-n)\,n/L + (1-c)\,n\,n/L$

Equilibrium: Bernoulli distrib

$P^{eq}(n) = C_L^n\, c^n (1-c)^{L-n}$

$\Psi_K(s)$ max of operator $\mathbb{W}_K$

$$\left(\mathbb{W}_K\right)_{n,n'} = e^{-s}\, W(n+1 \to n)\, \delta_{n',n+1} + e^{-s} W(n-1 \to n)\, \delta_{n',n-1} - r(n)\,\delta_{nn'}$$

It happen that $\underbrace{(P^{eq})^{-1/2}\, \mathbb{W}_K\, (P^{eq})^{1/2}}_{\equiv \mathbb{W}_K^{sym}}$ is symmetric (detailed below) also works for $\mathbb{W}_K$.

$$\boxed{\left(\mathbb{W}_K^{sym}\right)_{nn'} = e^{-s}\left[W(n+1 \to n)\, W(n \to n+1)\right]^{1/2} \delta_{n',n+1} + e^{-s}\left[W(n-1 \to n)\, W(n \to n-1)\right]^{1/2} \delta_{n',n-1} - r(n)\,\delta_{nn'}}$$

One uses:

$$\boxed{\max_{|p\rangle} \frac{\langle p|\, \mathbb{W}_K^{sym}\, |p\rangle}{\langle p|p\rangle} = \Psi_K(s)}$$

Courant-Fischer

One restrict ourselfs to $|p\rangle$ = ~~gaussian of mean density ρ~~ $p(n) = e^{L f(\rho)}$. One has

$$\frac{\langle p|\mathbb{W}_K^{sym}|p\rangle}{\langle p|p\rangle} = 2\, e^{-s}\sqrt{\rho^3(1-\rho)}\,\sqrt{c(1-c)} - c\rho(1-\rho) + (1-c)\rho^2$$

$$\max_{\rho} \frac{\sum_n e^{-s}(W^+W^-)^{1/2}\left(e^{f'} + e^{-f'}\right) - W^+ - W^- \; e^{2L f(\rho)}}{\sum_n e^{2N f(\rho)}}$$

Hence: $\frac{1}{L}\Psi_K(s) = -\min_{\rho} \mathcal{F}_K(\rho,s)$

$$\mathcal{F}_K(\rho,s) = \begin{cases} \rho\left\{2e^{-s}\sqrt{c(1-c)\rho(1-\rho)} - \left[c(1-\rho) + (1-c)\rho\right]\right\} & \text{constrained} \\ 2e^{-s}\sqrt{c(1-c)\rho(1-\rho)} - [\longrightarrow] & \text{unconstrained} \end{cases}$$

check: $s=0$; $\rho = c$ for the max, $\Psi_K = 0$ as expected

Other method: coherent states. Doombos V.

3.3 Dynamical "Landau" Free Energy Landscape :

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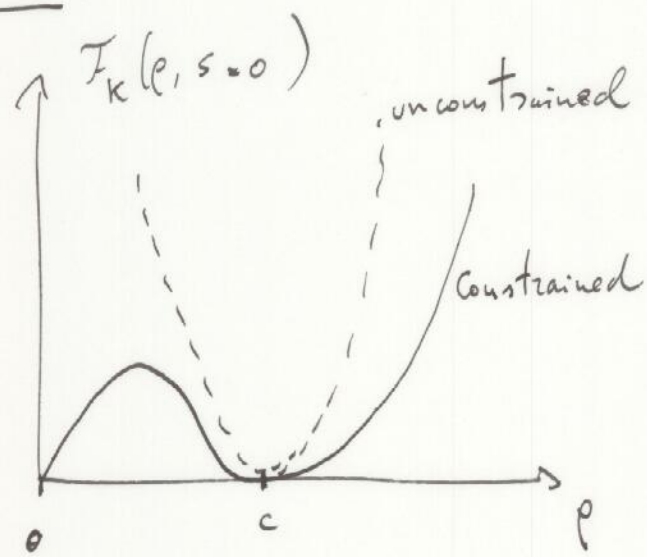
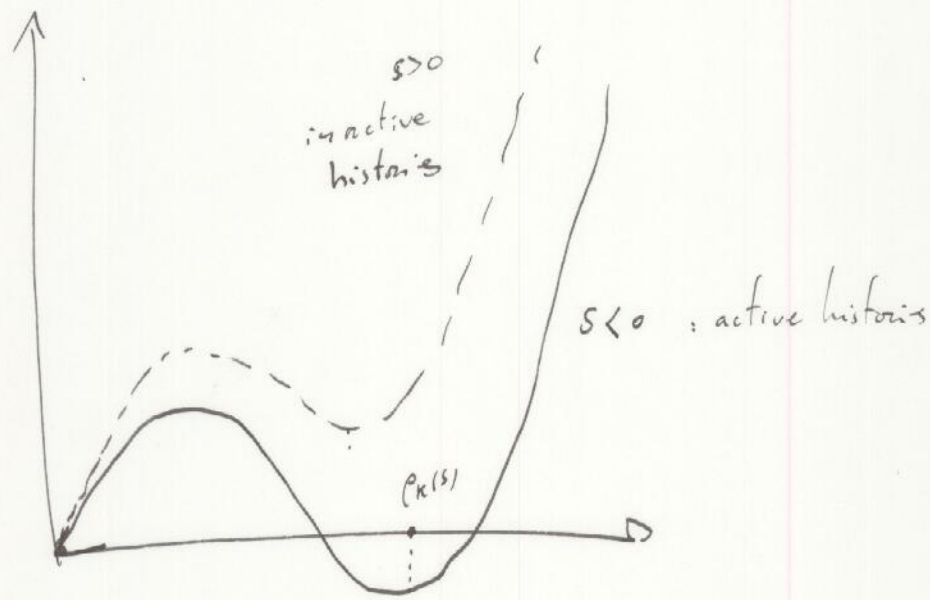
- Graphs :





$s=0$: steady state: lies on the critical line btw active and inactive histories

1^{st} order like transition

(dynamical) phase coexistence

order parameter $\rho_K(s) = \langle \int_0^t \text{density} \rangle$

- Physical meaning : one can show (eg with action for stoch. proc. or better with Donsker Varadhan)

$\text{Prob}\left(\begin{smallmatrix}\text{history with the}\\ \text{time average}\end{smallmatrix} ; s\right) \sim \exp\{L t \mathcal{F}_K(\rho_s)\}$

rq : static Landau FE is the same for

4 - Numerical results:

4.1 Method: (with J. Tailleur)

$$\begin{cases} W_s(\mathcal{C}\to\mathcal{C}') = e^{-s}\, W(\mathcal{C}\to\mathcal{C}') \\ r_s(\mathcal{C}) = \sum_{\mathcal{C}'} W_s(\mathcal{C}\to\mathcal{C}') \end{cases}$$

* let's rewrite the evolution:

$$\partial_t \hat{P}(\mathcal{C},s,t) = \underbrace{\sum_{\mathcal{C}'} W_s(\mathcal{C}'\to\mathcal{C})\,\hat{P}(\mathcal{C}',s,t) - r_s(\mathcal{C})\,\hat{P}(\mathcal{C},t)}_{\text{stochastic evolution with modified rates } W_s} + \underbrace{\left(r_s(\mathcal{C}) - r_s(\mathcal{C})\right)\hat{P}}_{\text{loss / gain of probability}}$$

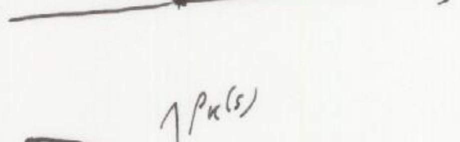
* Evolve large number of copies of the system in discrete (GKP) / continuous (VL, JT) time

At each step:
- normal evolution with rates W_s (for each clone)
- cloning / pruning of clones.

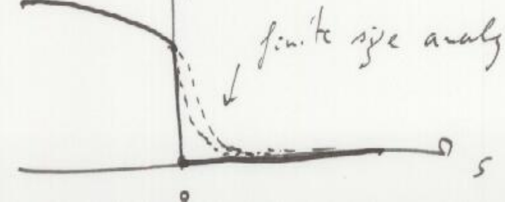
* $\psi_K(s)$ is the rate of increase / decrease of the population
  | (or of the rate at which one has to "rescale" the populat° to keep it constant – which one does).

* $\rho_K(s)$: mean density on $[0,t]$ window of time.

4.2 - Results: no access to $\mathcal{F}_K(\rho,s)$, but:
| $\psi_K(s)$
| $\rho_K(s)$ → order parameter



this confirm the scenario predicted in mean field —



5_ Conclusion, Perspectives :

. Quantitative descript° of phase coexistence:

. link with 4-point correlat° funct^s & dynamical growing lengths
. both analytical (mean 2d) & numerical approaches.

linked to ldf more in general.

. Questions & Perspectives :

. How universal is this ?
. "Real glasses" ? Lennard Jones
. Experiments ?

ref

. Main ending message :

- Glassy Phenomena take place before reaching steady state
- Our analysis ($s>0$, $s<0$) is done at infinite time.
("out of criticality")
- Handwaving interpretation:

At $s=0$: the system hesitates btw different kinds of histories. ($t\to\infty$ difficult to reach)
with s (suitably chosen — this depends on the physics of the system) one "isolates" these possibilities at $t\to\infty$.