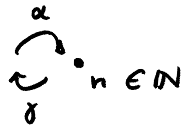
# Correction of the exam



## 1- Mapping between equilibrium and non-equilibrium

### 1-1 Contact with a reservoir




creation annihilation escape rate

a ... d is done in the lecture. One has $\mathbb{W} = \alpha a^{+} + \gamma a - (\alpha + \gamma \hat{n})$

For $s \leftrightarrow K$ = total activity $\mathbb{W}_s = e^{-s}(\alpha a^{+} + \gamma a) - (\alpha + \gamma \hat{n})$

$K \mapsto K+1$ at each "jump".

e. $|P_\rho\rangle = \sum_{n\geq 0} e^{-\rho} \frac{\rho^n}{n!} |n\rangle$ vector associated to Poisson distribution

- $a|P_\rho\rangle = \sum_{n\geq 0} e^{-\rho} \frac{\rho^n}{(n-1)!} |n-1\rangle = \sum_{n\geq 0} e^{-\rho} \frac{\rho^{n+1}}{n!} |n\rangle \Rightarrow \boxed{a|P_\rho\rangle = \rho |P_\rho\rangle}$
- $a^{+}|P_\rho\rangle = \sum_{n\geq 0} e^{-\rho} \frac{\rho^n}{n!} |n+1\rangle = \sum_{n\geq 0} e^{-\rho} \frac{\rho^{n-1}}{n!} \underbrace{n|n\rangle}_{\hat{n}|n\rangle} \Rightarrow \boxed{a^{+}|P_\rho\rangle = \frac{\hat{n}}{\rho} |P_\rho\rangle}$

f- $\mathbb{W}|P_\rho\rangle = \left\{\left(\alpha \frac{\hat{n}}{\rho} - \gamma \hat{n}\right) + \gamma\rho - \alpha\right\} |P_\rho\rangle \Rightarrow$ $\underline{\text{for } \rho = \frac{\alpha}{\gamma}, \ \mathbb{W}|P_\rho\rangle = 0}$

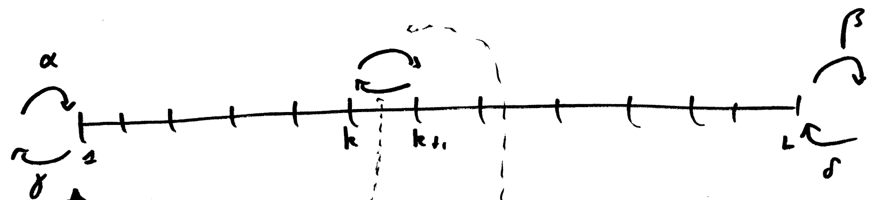
$|P_\rho\rangle$ is the equilibrium distribution, for $\rho = \frac{\alpha}{\gamma}$

g- $\langle n\rangle = \sum_n n P_\rho(n) = \sum_n e^{-\rho} \frac{\rho^n}{(n+1)!} = \rho \underbrace{\left(\sum_n \frac{\rho^n}{n!}\right)}_{e^\rho} . e^{-\rho} \Rightarrow \boxed{\langle n\rangle = \rho}$

$\rho$ is the average density

h- The site is '<u>in equilibrium</u>' with a reservoir of the same density $\rho = \frac{\alpha}{\gamma}$.

### 1.2 A chain in contact with reservoirs




a- $$\mathbb{W} = \underbrace{\begin{matrix}\alpha a_1^{+} + \gamma a_1 \\ - (\alpha + \gamma \hat{n}_1)\end{matrix}}_{\text{contact as in 1.1}} + \sum_{k=1}^{L} \begin{matrix} a_k^{+} a_{k+1} + a_{k+1}^{+} a_k \\ - (\hat{n}_{k+1} + \hat{n}_k) \end{matrix} + \underbrace{\begin{matrix}\delta a_L^{+} + \beta a_L \\ - (\delta + \beta \hat{n}_L)\end{matrix}}_{\text{contact as in 1.1}}$$

b- One expect the system to be in <u>equilibrium</u> if densities at boundaries are the same, i.e iff $\boxed{\frac{\alpha}{\gamma} = \frac{\delta}{\beta}}$

1.2 Correction exam

12c-. terms describing the contact with reservoirs verify

$$\mathbb{W}_{boundaries}|\rho\cdots\rho\rangle = 0 \qquad \text{since} \quad \rho = \frac{\alpha}{\gamma} = \frac{\delta}{\beta}$$

. terms from the bulk, from 1.1.e, write as:

$$\left(a_k^\dagger a_{k+1} + a_{k+1}^\dagger a_k - \hat n_{k+1} - \hat n_k\right)|\rho\cdots\rho\rangle = \underbrace{\left(\frac{\hat n_k}{\rho}\rho + \frac{\hat n_{k+1}}{\rho}\rho - \hat n_{k+1} - \hat n_k\right)}_{=0}|\rho\cdots\rho\rangle$$

Hence $|\rho\cdots\rho\rangle$ is the steady state.

d- if $\frac{\alpha}{\gamma} \neq \frac{\delta}{\beta}$ then one cannot have the same $\rho$ everywhere.

e- For a steady state of the form $|\rho_1\cdots\rho_L\rangle$ one may collect terms of same $\hat n_k$ with use again of 1.1.e to find the action of $a_k^\dagger$, $a_k$:

$$\mathbb{W}|\rho_1\cdots\rho_L\rangle = \Big\{\underbrace{\left(\frac{\alpha}{\rho_1} - \gamma + \frac{\rho_2}{\rho_1} - 1\right)}\hat n_1 + \sum_{k=2}^{L-1}\underbrace{\left(\frac{\rho_{k+1}+\rho_{k-1}}{\rho_k} - 2\right)}\hat n_k + \underbrace{\left(\frac{\delta}{\rho_L} - \beta + \frac{\rho_{L-1}}{\rho_L} - 1\right)}\hat n_L\Big\}|\rho_1\cdots\rho_L\rangle$$

$$+\underbrace{\Big\{\gamma\rho_1 - \alpha + \beta\rho_L - \delta\Big\}}|\rho_1\cdots\rho_L\rangle \quad \leftarrow \text{constant term.}$$

a sufficient condition is to cancel those terms.

f- the bulk equation $\frac{\rho_{k+1}+\rho_{k-1}-2\rho_k}{\rho_k} = 0$ imposes $\rho_k = A(k-1) + B$

The solution to the boundary equations writes: denoting $\rho_r = \frac{\delta}{\beta}$ $\rho_l = \frac{\alpha}{\gamma}$

$$A = \frac{\delta/\beta - \alpha/\gamma}{L-1+\frac{1}{\beta}+\frac{1}{\gamma}} = \frac{\rho_r - \rho_l}{L-1+\frac{1}{\beta}+\frac{1}{\gamma}}$$

$$B = \underbrace{\frac{\alpha}{\gamma}}_{=\rho_l} + \frac{1}{\gamma}\cdot\frac{\rho_r-\rho_l}{L-1+\frac{1}{\beta}+\frac{1}{\gamma}}$$

g- $\langle n_{k+1} - n_k\rangle = \rho_{k+1} - \rho_k = A = \frac{\rho_r-\rho_l}{L-1+\frac{1}{\beta}+\frac{1}{\gamma}}$ flow: $=0$ iff $\rho_r = \rho_l$ iff equilibrium; $\neq 0$ iff non-equilibrium

Note: posing $x = k/L$ $0 \le x \le 1$ continuous variable, one has

$$\rho(x) = \rho_{k/L} = (\rho_r - \rho_l)x + \rho_r \qquad \text{linear simple profile.}$$

The equilibrium is not here: there is a steady non-zero flow of particles.

## 1.3 - Fluctuation of current - Mapping eq ↔ non-eq

1.3 Exam correction

a - $Q \mapsto Q \begin{smallmatrix}+1\\-1\end{smallmatrix}$ Hence

$$\mathbb{W}_s = \alpha e^{-s} a_1^\dagger + \gamma e^{s} a_1 + \sum_{k=1}^{L-1}\left[e^{-s} a_{k+1}^\dagger a_k + e^{s} a_k^\dagger a_{k+1}\right] + \beta e^{-s} a_L + \delta e^{s} a_L^\dagger$$
$$- \left\{\alpha + \gamma \hat{n}_1 + \sum_{k=1}^{L-1}\left[\hat{n}_k + \hat{n}_{k+1}\right] + \beta \hat{n}_L + \delta\right\}$$

b - $Q$ is given p. 7.7 of the lecture. $a_k^\dagger \mapsto z^k a_k^\dagger$
$a_k \mapsto z^{-k} a_k$

Hence: $\begin{cases} a_{k+1}^\dagger a_k \mapsto z\, a_{k+1}^\dagger a_k \\ a_k^\dagger a_{k+1} \mapsto z^{-1} a_k^\dagger a_{k+1} \end{cases}$ and, taking $\boxed{z = e^{-s}}$ one obtains

$$\mathbb{W}_s \simeq \alpha a_1^\dagger + \gamma a_1 + \sum_{k=1}^{L-1}\left[a_{k+1}^\dagger a_k + a_k^\dagger a_{k+1}\right] + \beta e^{-(L+1)s} a_L + \delta e^{+(L+1)s} a_L^\dagger$$
$$- \{\ \text{same}\ \}$$

c - $\mathbb{W}_s = \mathbb{W}^1_{(L+1)s}$ with $\mathbb{W}^1_s$ for $s \leftrightarrow$ current through the contact to the right reservoir.

using that $\psi$ is the maximum eigenvalue:

$$\boxed{\psi_{tot}(s) = \psi_1((L+1)s)}$$

$L+1$ represents the total number of bonds (including those with the reservoirs).

d - One performs $\begin{cases} a_k^\dagger \mapsto a_k^\dagger + x \\ a_k \mapsto a_k \end{cases}$ $x \in \mathbb{R}$ $\Bigg\} \Rightarrow$ bulk $= \sum_{k=1}^{L-1} -(a_{k+1}^\dagger - a_k^\dagger)(a_{k+1} - a_k)$ is inchanged

On boundaries, using $\hat{n}_1 \mapsto (a_1^\dagger + x) a_1 = \hat{n}_1 + x a_1$ one gets:
$\hat{n}_L \mapsto (a_L^\dagger + x) a_L = \hat{n}_L + x a_L$

$$\mathbb{W}_s \simeq \overbrace{x(\alpha + \delta e^{(L+1)s})}^{C_0} + \left\{\alpha a_1^\dagger + \gamma(1-x) a_1\right\} + \mathbb{W}_{bulk} + \left\{\beta(e^{-(L+1)s} - x) a_L + \delta e^{(L+1)s} a_L^\dagger\right\}$$
$$- \{\alpha + \gamma \hat{n}_1\} - \{\beta \hat{n}_L + \delta\}$$

e - One now performs $\begin{cases} a_k^\dagger \mapsto z\, a_k^\dagger \\ a_k \mapsto z^{-1} a_k \end{cases}$ again $\mathbb{W}_{bulk}$ is not modified

1.4 Exact construction

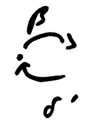
To interpret the term in $\alpha, \gamma$ as a contact with a reservoir, one choses $z = (1-x)$ since it indeed rewrites

$$\mathbb{W}_{left} \doteq \overset{\text{from } \mathcal{E}_0}{x\alpha} + \Big\{ \overbrace{\alpha(1-x)}^{\equiv \alpha'} a_1^+ + \gamma a_1 \Big\} \qquad \equiv \text{contact with reservoir}$$
$$\underset{\text{cancel!}}{-x\alpha} - \Big\{ \alpha(1-x) + \gamma \hat{n}_1 \Big\} \qquad \text{of density } \frac{\alpha}{\gamma}(1-x)$$

to compensate this modification

On the other hand, the term in $\beta, \delta$ rewrites:

$$\mathbb{W}_{right} \doteq \overset{\text{from } \mathcal{E}_0}{x\delta e^{(L+1)s}} + \Big\{ \delta e^{(L+1)s}(1-x) a_L^+ + \beta \Big(\frac{e^{-(L+1)s} - x}{1-x}\Big) a_L \Big\} - \Big\{ \delta + \beta \hat{n}_L \Big\} \qquad (*)$$

We now search for $\delta'$ and $s'$ such that it takes a physical form describing counting the current (with $s'$) for the contact with a reservoir 

$$\mathbb{W}_{right} \overset{?}{\doteq} \mathcal{E}_2 + \Big\{ \delta' e^{(L+1)s'} a_L^+ + \beta e^{-(L+1)s'} a_L \Big\} - \Big\{ \delta' + \beta \hat{n}_L \Big\}$$

We thus have the two equations:
$$\begin{cases} e^{-(L+1)s'} = \left(e^{-(L+1)s} - x\right)/(1-x) & (i) \\ \delta' e^{(L+1)s'} = \delta e^{(L+1)s}(1-x) & (ii) \end{cases}$$

The  describes contact with reservoir of density $\frac{\delta'}{\beta}$.

One fixes from (i) the value of $e^{-(L+1)s'} = \frac{e^{-(L+1)s} - x}{1-x}$ hence imposing from (ii):

$$\delta' = \delta . e^{(L+1)s} \left(e^{-(L+1)s} - x\right) \frac{1-x}{1-x} = \underline{\delta . \left(1 - e^{(L+1)s} x\right)}$$

We now fix $x$ from the equality of densities $\frac{\delta'}{\beta} = \frac{\alpha'}{\gamma}$ i.e. :

$$\boxed{\frac{\delta}{\beta}\left(1 - e^{(L+1)s} x\right) = \frac{\alpha}{\gamma}(1-x)}$$ This is the equation verified by $x$

Note that there is no solution for $s=0$ in general. Here, for $s \neq 0$, one finds

$$\boxed{x = \frac{\rho_\ell - \rho_r}{\rho_\ell - e^{(L+1)s}\rho_r}}$$ with again $\rho_\ell = \frac{\alpha}{\gamma}$ $\rho_r = \frac{\delta}{\beta}$.

(1-3.e, followed)

1.5 Correction exam

We deduce from this result that $\boxed{e^{-(L+1)s'} = \frac{\rho_l}{\rho_r} e^{-(L+1)s}}$ which fixes $s'$.

The constant $\mathcal{C}_2$ is such that:

$$x\,\delta e^{(L+1)s} - \delta = \mathcal{C}_2 - \delta' \quad \text{ie} \quad \mathcal{C}_2 = \underbrace{\delta' - \delta}_{=-\delta x e^{(L+1)s}} + \delta x e^{(L+1)s}$$

from the expression of $\delta'$

finally: $\boxed{\mathcal{C}_2 = 0}$

To summarize:

$$\mathbb{W}_s^{tot}\Big|_{\substack{\alpha\beta\\\gamma\delta}} \equiv \underbrace{\mathbb{W}^1_{(L+1)s'}\Big|_{\substack{\alpha'\beta\\\gamma\delta'}}}_{\text{contact with equilibrium reservoirs of density } \rho' = \frac{\delta'}{\beta} = \frac{\alpha'}{\gamma} = \frac{\alpha}{\gamma}(1-x)}$$

with

$$x = (\rho_l - \rho_r)/(\rho_l - e^{(L+1)s}\rho_r)$$

$$\alpha' = (1-x)\,\alpha$$

$$\delta' = \delta(1 - e^{(L+1)s}x)$$

$$e^{-(L+1)s'} = \frac{\rho_l}{\rho_r} e^{-(L+1)s}$$

Using the previous correspondence b/w $\mathbb{W}_s^1$ and $\mathbb{W}_s^{tot}$ one thus has

$$\boxed{\underbrace{\Psi_{tot}(s)\Big|_{\substack{\alpha\beta\\\gamma\delta}}}_{\text{non-equilibrium}} = \underbrace{\Psi_{tot}(s')\Big|_{\substack{\alpha'\beta\\\gamma\delta'}}}_{\substack{\text{contact with reservoirs}\\ \text{of same densities}\\ \Rightarrow \text{equilibrium fluctuations}}}}$$

as expected.

2 - <u>Bosonic MF. KCM</u>

2.1 Exam correction

$$W(n \to n+1) = c\, n \qquad \text{creation at rate } c$$

$$W(n \to n-1) = \underbrace{\tfrac{n}{L}}_{\text{counting}} \underbrace{(n-1)}_{\text{kinetic constraint}} \qquad \text{annihilation} - \frac{1}{L}$$

2.1.a -

b. using the generic expression of $\mathbb{W}_s$ for $s \to K$ and $n$-dependent rates:

$$\mathbb{W}_s = e^{-s}\left(c a^{+}\hat{n} + \tfrac{1}{L} a(\hat{n}-1)\right) - \left(c\hat{n} + \tfrac{1}{L}\hat{n}(\hat{n}-1)\right)$$

c - $(a a^{+} - a^{+} a)|n\rangle = a|n+1\rangle - n\, a^{+}|n-1\rangle = [(n+1) - n]|n\rangle \Rightarrow [a, a^{+}] = 1$

$a^{+}\hat{n}|n\rangle = a^{+}a^{+}a|n\rangle = a^{+}(a a^{+} - 1)|n\rangle = (a a^{+} - 1)a^{+}|n\rangle \Rightarrow \boxed{a^{+}\hat{n} = (\hat{n}-1)a^{+}}$

$a(\hat{n}-1)|n\rangle = (a a^{+} a - a)|n\rangle = (a^{+}a a + a - a)|\hat{n}\rangle = a^{+}a\, a|n\rangle \Rightarrow \boxed{a(\hat{n}-1) = \hat{n} a}$

d - $\mathbb{W}|P_\rho\rangle$ : $\mathbb{W} = \left(c(\hat{n}-1)a^{+} + \tfrac{1}{L}\hat{n} a\right) - \left(c\hat{n} + \tfrac{1}{L}\hat{n}(\hat{n}-1)\right)$

$$\mathbb{W}|P_\rho\rangle = \left[\left(c(\hat{n}-1)\frac{\hat{n}}{\rho} + \tfrac{1}{L}\hat{n}\rho\right) - \left(c\hat{n} + \tfrac{1}{L}\hat{n}(\hat{n}-1)\right)\right]|P_\rho\rangle$$

hence $\boxed{\rho = Lc}$ ensures $\mathbb{W}|P_\rho\rangle = 0$ $\quad |P_{\rho = Lc}\rangle$ steady state.

e - it takes a single Poisson form.
However for $s \neq 0$, one cannot find such a form.

2-2-a - see lecture

b - see 11d

c - $$S = \int_0^t d\tau \left[e^{-s}\left(c\hat{\varphi}^2\varphi + \tfrac{1}{L}\hat{\varphi}\varphi^2\right) - \left(c\hat{\varphi}\varphi + \tfrac{1}{L}\hat{\varphi}^2\varphi^2\right)\right]$$

d - <u>Cole-Hopf</u> $\varphi = L\rho e^{-\hat{\rho}}$ $\quad \hat{\varphi} = e^{\hat{\rho}}$

$$\underline{J_s(\rho,\hat{\rho}) = -(c+\rho)\rho + e^{-s}\left(c e^{\hat{\rho}} + \rho e^{-\hat{\rho}}\right)\rho}$$

e. <u>Saddle point equations</u>:

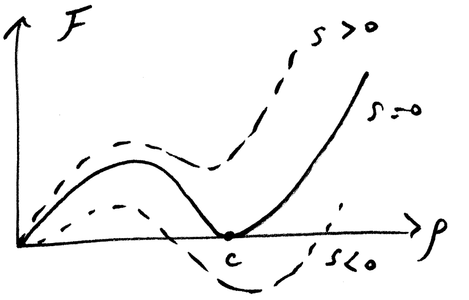
$$0 = \frac{\partial J_s}{\partial \rho} = c\left(e^{-s}e^{\hat{\rho}} - 1\right) + 2\left(e^{-s}e^{-\hat{\rho}} - 1\right)\rho$$

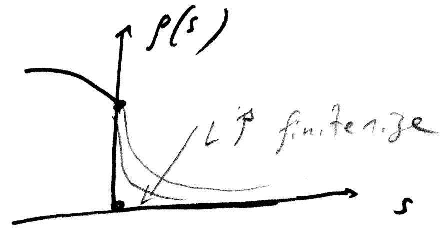
$$0 = \frac{\partial J_s}{\partial \hat{\rho}} = e^{-\hat{\rho}}e^{-s}\left(c e^{2\hat{\rho}} - \rho\right)\rho \Rightarrow \boxed{e^{\hat{\rho}} = \sqrt{\rho/c}}$$

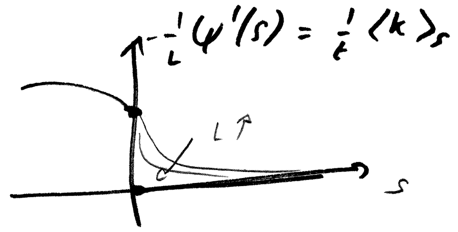
f - $$F(\rho,s) = -J_s(\rho,\hat{\rho}) \Rightarrow \boxed{F(\rho,s) = \rho(c+\rho) - 2e^{-s}\rho\sqrt{c\rho}}$$

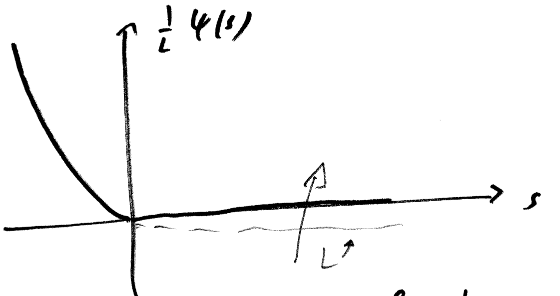
(2.2 correction exam

2.3.




See the lecture for the interpretation.

Exercice 3 - $K$ & $R = \int r$   $\langle e^{-sK-\sigma R}\rangle \sim e^{t\Psi(s,\sigma)}$   (3.1 Condo exam)

3-1-a $R = \int_0^t d\tau\, r(\mathcal{C}(\tau))$ of kind "$A_2$" $\Rightarrow$ contribution diagonal in the evolution operator

$$(\mathbb{W}_{s\sigma})_{\mathcal{C}\mathcal{C}'} = \underbrace{e^{-s}}_{K \mapsto K+1} W(\mathcal{C}' \to \mathcal{C}) - r(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'} \quad \underbrace{-\sigma r(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'}}_{R = \int r}$$

b- $$(\mathbb{W}_{s,\sigma})_{\mathcal{C},\mathcal{C}'} = (1+\sigma)\Big[\underbrace{\frac{e^{-s}}{1+\sigma}}_{= e^{-(s+\log(1+\sigma))}} W(\mathcal{C}' \to \mathcal{C}) - r(\mathcal{C})\Big] = (1+\sigma)\left(\mathbb{W}_{s+\log(1+\sigma),0}\right)_{\mathcal{C}\mathcal{C}'}$$

c- By max eigenvalue: $\Psi(s,\sigma) = \Psi(s+\log(1+\sigma),0)$ $(\star)$

d- $$-\partial_s \Psi(s,\sigma)\Big|_{\sigma=0} = \frac{1}{t}\frac{\langle K e^{-sK}\rangle}{\langle e^{-sK}\rangle} = \frac{1}{t}\langle K\rangle_s$$

$$-\partial_\sigma \Psi(s,\sigma)\Big|_{\sigma=0} = \frac{1}{t}\frac{\langle R e^{-sK}\rangle}{\langle e^{-sK}\rangle} = \frac{1}{t}\langle R\rangle_s$$

e- one has: $$-\partial_s \Psi(s,\sigma)\Big|_{\sigma=0} = -\partial_s\left[\Psi(s+\log(1+\sigma),0)\right]\Big|_{\sigma=0} = \frac{1}{t}\langle K\rangle_s$$

$$-\partial_\sigma \Psi(s,\sigma)\Big|_{\sigma=0} = -\left[\partial_\sigma(\text{---})\right]\Big|_{\sigma=0} = -\left[(\partial_s\Psi)\cdot\frac{1}{1+\sigma}\right]\Big|_{\sigma=0}$$

Differentiation $(\star)$ wrt $\sigma\big|_{\sigma=0}$: $\frac{1}{t}\langle K\rangle = \Psi(s) + \frac{1}{t}\langle R\rangle$

finally $$\boxed{\Psi(s) = \frac{1}{t}\langle K\rangle_s - \frac{1}{t}\langle R\rangle_s}$$

3.2 - $$(\mathbb{W}_{s\sigma})_{\mathcal{C}\mathcal{C}'} = \underbrace{e^{-s}}_{\text{time changed by a factor } e^{-s}}\Big(\underbrace{W(\mathcal{C}'\to\mathcal{C}) - \delta_{\mathcal{C}\mathcal{C}'}r(\mathcal{C})}_{\text{dynamics is not modified}} - \underbrace{\left[e^{s}(1+\sigma)-1\right]r(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'}}_{\text{cloning rate}}\Big)$$

see lecture for the implementation