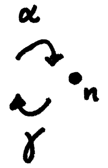
Exercice 1 : Mapping between equilibrium and non-equilibrium

1-1- Describing the contact with a reservoir -



consider a site with $n = 0, 1, \dots$ particles

particles can go onto the site at rate α

exit from — — γ .

1.1-a- **Justify** that the probability distribution $P(n,t)$ evolves with

$$\partial_t P(n,t) = \alpha P(n-1,t) + \gamma(n+1) P(n+1,t) - (\alpha + \gamma n) P(n,t)$$

b- We introduce states $|n\rangle$, operators a, a^+ with $a|n\rangle = n|n-1\rangle$, $a^+|n\rangle = |n+1\rangle$

And $|P(t)\rangle = \sum_n P(n,t)|n\rangle$

Show in details that the evolution operator $\mathbb{W}$ such that $\partial_t |P(t)\rangle = \mathbb{W}|P\rangle$ is:

$$\mathbb{W} = \alpha a^+ + \gamma a - (\alpha + \gamma \hat{n}) \quad \text{with} \quad \hat{n} = a^+ a$$

c- **Interpret** each of the terms of this operator.

d- We now take s conjugated to K = total activity, and introduce $\hat{P}(n,s,t) = \sum_K P(n,K,t) e^{-sK}$.

Write (you don't need to derive the result) the operator $\mathbb{W}_s$ such that $\partial_t |\hat{P}(s,t)\rangle = \mathbb{W}_s |\hat{P}(s,t)\rangle$.

e- Consider a Poisson distribution $P_\rho(n) = e^{-\rho} \frac{\rho^n}{n!}$, and $|P_\rho\rangle = \sum_{n\geq 0} P_\rho(n)|n\rangle$.

Compute $a|P_\rho\rangle$, and relate $a^+|P_\rho\rangle$ to $\hat{n}|P_\rho\rangle$.

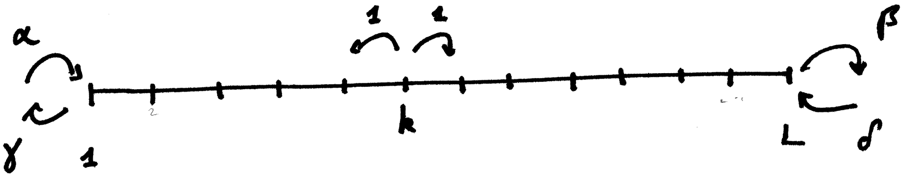
f- **Find** ρ such that $\mathbb{W}|P_\rho\rangle = 0$.

g- **Compute** $\langle n \rangle = \sum_n n \, P_\rho(n)$

Reminders:
$n! = n(n-1)!$
$\sum_{n\geq 0} \rho^n/n! = e^\rho$

h- **Justify** why one can say that « the site is in contact with a reservoir of particle density $\rho = \frac{\alpha}{\gamma}$ » ?

1.2 - A chain in contact with two reservoirs -



We now consider a system of L sites (k = 1 … L) in contact with two reservoirs (see the figure above). Bulk jump rates are equal to 1.

1.2 - a - Justify in full details (but without writing the Markov equation) that

$$\mathbb{W} = \alpha a_1^{+} + \gamma a_1 + \sum_{k=1}^{L-1}\left[a_k^{+} a_{k+1} + a_{k+1}^{+} a_k\right] + \delta a_L^{+} + \beta a_L$$
$$- \left\{\alpha + \gamma \hat{n}_1 + \sum_{k=1}^{L-1}\left[\hat{n}_{k+1} + \hat{n}_k\right] + \delta + \beta \hat{n}_L\right\}$$

[use 1-1-b]

b - What condition do you expect to be required for the system to be in equilibrium ? (Don't do any computation; condition is on $\frac{\alpha\beta}{\gamma\delta}$)

c - We denote by $|\rho_1, \cdots, \rho_L\rangle$ the state corresponding to the product of Poisson distributions, at every site k of parameter ρ_k :

$$|\rho_1, \cdots, \rho_L\rangle = \sum_{\vec{n}} P_{\rho_1}(n_1) \cdots \underbrace{P_{\rho_L}}_{\text{defined in 1.1-e}}(n_L)\, |n_1 \ldots n_L\rangle$$

Note that a_k and a_k^{+} act only on site k.

Using 1-1-e, show that, if $\frac{\alpha}{\gamma} = \frac{\delta}{\beta} = \rho$, then $|P_{eq}\rangle = |\rho, \cdots, \rho\rangle$ is the steady state (i.e. $\mathbb{W}|P_{eq}\rangle = 0$).

d - What happens if $\frac{\delta}{\beta} \neq \frac{\alpha}{\gamma}$? [i.e. is the steady-state simple?]

e - We search for a steady-state of the form $|P_{st}\rangle = |\rho_1, \cdots, \rho_L\rangle$.

Show that the condition for the ρ_k's are :

$\underline{k=1}$: $\rho_1^{-1}\alpha - \gamma + \frac{\rho_2}{\rho_1} - 1 = 0$ $\underline{k=L}$: $\rho_L^{-1}\delta - \beta + \frac{\rho_{L-1}}{\rho_L} - 1 = 0$

$\underline{2 \leq k \leq L-1}$: $\frac{\rho_{k+1} + \rho_{k-1}}{\rho_k} - 2 = 0$ {Hint : consider the coefficients of every $\hat{n}_k$}

(1.3

Exam

1-2-f- <u>Find</u> a solution of the form $p_k = A(k-1) + B$

g- <u>Determine</u> the mean flow of particle $\langle n_{k+1} - n_k \rangle$. Comment!

<u>Justify</u> that you have found a <u>non-equilibrium</u> steady state

<u>1.3- Fluctuations of current ; mapping between equilibrium and non-equilibrium</u> -

1.3. a- <u>Justify</u> that for s conjugated to the total current Q

$$Q = \#(\text{jumps } \curvearrowright) - \#(\text{jumps } \curvearrowleft) \qquad \left[\begin{array}{l}\text{including those}\\ \text{at the boundaries}\end{array}\right]$$

one has :

$$\mathbb{W}_s = \alpha e^{-s} a_1^+ + \gamma e^{s} a_1 + \sum_{k=1}^{L-1} \left[e^{-s} a_{k+1}^+ a_k + e^{s} a_k^+ a_{k+1} \right] + \beta e^{-s} a_L + \delta e^{s} a_L^+$$

$$- \left\{ \text{same expression as for } \mathbb{W} \text{ in 1.2.a} \right\}$$

b- <u>Using results</u> of p.3.7 of the lecture, find a transformation such that

$$\begin{cases} Q^{-1} a_k^+ Q = z^{k} a_k^+ \\ Q^{-1} a_k Q = z^{-k} a_k \end{cases}$$

<u>Chose</u> z (what is it's value?) such that

$$Q^{-1} \mathbb{W}_s Q = \begin{pmatrix}\text{same terms as } \mathbb{W}\\ \text{for } \alpha, \gamma\end{pmatrix} + \begin{pmatrix}\text{same terms as } \mathbb{W}\\ \text{for } 1 \leq k \leq L-1\end{pmatrix} + \begin{array}{l} \beta e^{-(L+1)s} a_L + \delta e^{(L+1)s} a_L^+ \\ - \beta \hat{n}_L - \delta \end{array}$$

c. Consider the l.d.f $\Psi_1(s)$ for s conjugated to the current to the reservoir at the ~~left~~ right.

$$\Psi_{tot}(s) = \max \operatorname{Sp} \mathbb{W}_s \qquad (\text{total current } Q)$$

<u>Justify</u> that : $\Psi_{tot}(s) = \Psi_1((L+1)s)$

What does the "$L+1$" represents in this identity?

1.3.d- Note that the bulk term (the Σ) in $\mathbb{W}$ (i.e. @$s=0$) rewrites

(1.4

Exam

$$-\sum_{k=1}^{L-1}(a^+_{k+1}-a^+_k)(a_{k+1}-a_k) \equiv \mathbb{W}_{bulk}$$

Using again a similarity transformation of p. 3.7 of the lecture (<u>which one</u>?) to perform

$$\left| \begin{array}{lll} Q^{-1}\, a^+_k\, Q & = a^+_k + x & \quad x \in \mathbb{R} \\ Q^{-1}\, a_k\, Q & = a_k & \end{array} \right.$$

show that [the symbol $\hat{=}$ means that spectrum of both side are the same]

$$\mathbb{W}_s \,\hat{=}\, \underset{\substack{\uparrow \\ \text{constant to find}}}{\mathcal{C}_0} + \alpha a^+_1 + \gamma(1-x)a_1 + \mathbb{W}_{bulk} + \delta a^+_L e^{+(L+1)s} + \beta(e^{-(L+1)s} - x)a_L$$

$$-\{\alpha + \gamma\, \hat{n}_1\} \qquad\qquad - \delta \qquad - \beta \quad \hat{n}_L$$

1.3.e [DIFFICULT!]

Using a (in fact, several) similarity transformation similar to that of 1.3.6 show that

$$\boxed{\underbrace{\Psi_{tot}(s)}_{\text{initial one}} = \underbrace{\mathcal{C}_1}_{\text{constant}} + \underbrace{\Psi_{tot}(s'; eq)}_{\text{for a system with reservoirs of same density at boundaries}}}$$

<u>Hints</u>: ① consider $\left\{ \begin{array}{l} a^+_k \mapsto Z\, a^+_k \\ a_k \mapsto Z^{-1} a_k \end{array} \right.$ with Z adjusted so that terms in α, γ can be interpreted, <u>up to a constant</u>, as describing a contact with a reservoir of rates $\overset{\alpha'}{\curvearrowright}\bullet\underset{\gamma}{\curvearrowleft}$

(one has $Z = 1-x$; $\alpha' = Z\alpha$)

② adjust x such that terms at site L can be interpreted as describing contact with a reservoir of same density as at the left (one has $\frac{\alpha'}{\gamma} = \frac{\delta}{\beta}$)

③ interpret the terms in a^+_L, a_L as counting some current for a value of s' to be determined.

Exercice 2 A bosonic mean-field KCM

2.1

Exam

We consider the facilitated model with rates

$$W(n \to n+1) = c \cdot n$$

$$W(n \to n-1) = \frac{n}{L} \cdot \underbrace{(n-1)}$$

kinetic constraint
(= # active neighbours

L being a large number

n is a number of active sites

2.1 - Operator of evolution

2.1.a - What is the interpretation of the rates above?

b- Show that: for $s \leftrightarrow K$

$$\mathbb{W}_s = e^{-s}\left\{ c a^+ \hat{n} + \frac{1}{L} a (\hat{n}-1) \right\} - \left\{ c \hat{n} + \frac{1}{L} \hat{n} (\hat{n}-1) \right\}$$

c- Reminders on a^+, a, and properties

$a|n\rangle = n|n-1\rangle$
$a^+|n\rangle = |n+1\rangle$

Show that $a a^+ - a^+ a = 1$ using ↗

Infer from that property the following ones:

$$a^+ \hat{n} = (\hat{n}-1) a^+$$

$$a(\hat{n}-1) = \hat{n} a$$

d- Steady state: search for a steady-state of $\mathbb{W} = \mathbb{W}_{s=0}$ of the Poisson form $|P_\rho\rangle$ (see 1.1.e, 1.1.f)
[use 2.1.c] One finds $\boxed{\rho = Lc}$.

e- Is the steady-state simple?
For $s=0$, can you find a simple eigenvector?

2.2

Exam

2.2 - Action :

2.2.a - Describe in a few lines how to obtain

$$\langle e^{-sK}\rangle = \int \mathcal{D}\varphi\, \mathcal{D}\hat{\varphi}\; e^{-S[\varphi,\hat{\varphi}]} \qquad \Big| \begin{array}{l} \varphi = \varphi(t) \\ \hat{\varphi} = \hat{\varphi}(t) \end{array}$$

$$S[\varphi,\hat{\varphi}] = \int_0^t d\tau \left\{ \hat{\varphi}\, \partial_\tau \varphi - \mathcal{H}_s[\hat{\varphi},\varphi] \right\}$$

$\mathcal{H}_s[\hat{\varphi},\varphi]$ obtained from $\underbrace{\mathbb{W}_s}_{\text{normal-ordered}}$ with $\begin{array}{l} a \mapsto \varphi \\ a^+ \mapsto \hat{\varphi} \end{array}$

b - Show that the normal-ordered form of $\mathbb{W}_s$ is

$$\mathbb{W}_s = e^{-s}\left\{ c\, a^{+2} a + \frac{1}{L} a^{+} a^{2} \right\} - \left\{ c\, a^{+} a + \frac{1}{L} a^{+2} a^{2} \right\}$$

c - Write the corresponding $S[\varphi, \hat{\varphi}]$

d - One assumes that the most relevant $\varphi, \hat{\varphi}$ are constant in time

One evaluates $\int d\varphi\, d\hat{\varphi}\; e^{\mathcal{H}_s(\varphi,\hat{\varphi}).t}$ using saddle point

constant

approximation : the optimal $\varphi, \hat{\varphi}$ verify

$$\frac{\partial \mathcal{H}_s}{\partial \varphi} = 0 \qquad \frac{\partial \mathcal{H}_s}{\partial \hat{\varphi}} = 0$$

A better choice is to make a Cole-Hopf transform :

$$\begin{cases} \varphi = L\rho\, e^{-\hat{\rho}} \\ \hat{\varphi} = e^{\hat{\rho}} \end{cases}$$

What does represent $\hat{\varphi}\varphi$?

One sets $J_s(\rho,\hat{\rho}) = \frac{1}{L} \mathcal{H}_s(\varphi,\hat{\varphi})$

Compute $J_s(\rho,\hat{\rho})$

e - Write the saddle point equation $\frac{\partial J_s}{\partial \rho} = 0 \quad \frac{\partial J_s}{\partial \hat{\rho}} = 0$

From the second one, obtain $e^{\hat{\rho}}$ as a function of ρ, c, e^{-s}.

f - Substituting into $J_s(\rho,\hat{\rho})$, show finally that

$$\frac{1}{L}\psi(s) = -\min_{\rho} \mathcal{F}(\rho,s), \quad \text{with} \quad \boxed{\mathcal{F}(\rho) = \rho(\rho+c) - 2\sqrt{c}\,\rho\, e^{-s}\sqrt{\rho}}$$

2.3 Exam

2 - 3 - Dynamical phase transition :

a - Represent $\mathcal{F}(\rho, s)$ for
$s = 0$
$s > 0$
$s < 0$

b - Where is the absolute minimum of $\mathcal{F}(\rho, s)$ reached in each of those situations ?
We call $\rho(s)$ this minimum

c - Represent schematically $\psi(s)$ $\rho(s)$ $-\psi'(s)$
Interpret physically .

d - Redraw those graph including finite L effects that you expect.

(3.1

Exam

Exercise 3 - A relation between fluctuations of K and fluctuations of $R = \int_0^t d\tau\, r(\mathcal{C})$

Settings : a system with discrete configurations $\mathcal{C}$

transitions rates $W(\mathcal{C} \to \mathcal{C}')$

escape rate $r(\mathcal{C}) = \sum_{\mathcal{C}'} W(\mathcal{C} \to \mathcal{C}')$

$$\left.\begin{array}{l}\text{Observables: activity } K \\ \text{integrated escape rate } R = \int_0^t d\tau\, r(\underset{\substack{\uparrow \\ \text{at time } \tau}}{\mathcal{C}})\end{array}\right\} \text{depend on history.}$$

3.1 - One considers the dual

$$P(\mathcal{C}, K, R, t) \qquad \begin{array}{ccc} s & \text{conjugated to} & K \\ \sigma & \text{———} & R \end{array}$$

$$\hat{P}(\mathcal{C}, s, \sigma, t) \swarrow$$

and one defines $\Psi(s,\sigma) = \lim_{t\to\infty} \frac{1}{t} \log \langle e^{-sK - \sigma R} \rangle$

3.1a - **Justify** why $\Psi(s,\sigma) = \max \mathrm{Sp}\, \mathbb{W}_{s,\sigma}$

with $(\mathbb{W}_{s,\sigma})_{\mathcal{C}\mathcal{C}'} = e^{-s} W(\mathcal{C}' \to \mathcal{C}) - (1+\sigma) r(\mathcal{C})\, \delta_{\mathcal{C}\mathcal{C}'}$

3.1,b - **Show** in details that:

$$\mathbb{W}_{s,\sigma} = (1+\sigma)\, \mathbb{W}_{s+\log(1+\sigma),0}$$

c - Why is it that $\Psi(s,\sigma) = (1+\sigma)\, \Psi(s+\log(1+\sigma), 0)$?

d - Show that $\underbrace{\frac{1}{t}\langle K \rangle_s}_{\swarrow} \overset{\uparrow}{\underset{t\to\infty}{=}} -\partial_s \Psi(s,\sigma)|_{\sigma=0} \qquad \frac{1}{t}\langle R \rangle_s = -\partial_\sigma \Psi(s,\sigma)|_{\sigma=0}$

histories only weighted by e^{-sK}

e - **Deduce** from c and d that: $\underbrace{\psi(s)}_{\Psi(s,\sigma=0)} = \underbrace{\frac{1}{t}\langle K \rangle_s - \frac{1}{t}\langle R \rangle_s}_{\langle \dots \rangle_s \,:\, \sigma=0}$

[LONG!]

3.2 - Rewrite $(\mathbb{W}_{s,\sigma})_{\mathcal{C},\mathcal{C}'} = e^{-s}\left(W(\mathcal{C}' \to \mathcal{C}) - e^{s}(1+\sigma) r(\mathcal{C})\, \delta_{\mathcal{C}\mathcal{C}'} \right)$

Describe in a rather detailed way a **cloning algorithm** which allows to compute $\Psi(s,\sigma)$, using this rewriting.