Functional derivatives: continuous and discrete musings

Questions

In the same way as, for a discrete set $\mathbf{F} = (F_i)$ indexed by integers i and an 'action' $\mathcal{S}[\mathbf{F}]$ defined as the sum of local terms of the form $\mathcal{L}(F, j)$

$$\mathcal{S}[\mathbf{F}] = \sum_j \mathcal{L}(F_j, j) \tag{1}$$

one has

$$\frac{\partial \mathcal{S}[\mathbf{F}]}{\partial F_i} = \partial_F \mathcal{L}(F_i, i) \tag{2}$$

we would like, for a function $F(y)$ of a continuous variable y and an action functional

$$\mathcal{S}[F] = \int dy\, \mathcal{L}(F(y), y) \tag{3}$$

to determine the dependence of $\mathcal{S}$ in F, as

$$\frac{\delta \mathcal{S}[F]}{\delta F(y)} = \partial_F \mathcal{L}(F(y), y)\,, \tag{4}$$

having in mind that subtleties are to appear when the 'Lagrangian' $\mathcal{L}$ depends on derivatives of F. Note: we work in a space of functions F where F and its derivatives at $\pm\infty$ are zero.

Continuous variables

(i) Definitions

To find a good definition of the *functional derivative* $\frac{\delta}{\delta F(y)}$ that ensures the rule (3,4) to be correct, let's remark that it implies

$$\frac{\delta F(y_1)}{\delta F(y_2)} = \frac{\delta}{\delta F(y_2)} \int dy\, \delta(y - y_1) F(y) \quad \text{hence} \quad \boxed{\frac{\delta F(y_1)}{\delta F(y_2)} = \delta(y_2 - y_1)} \tag{5}$$

Conversely, if this rule is taken as a definition, and if the operator $\frac{\delta}{\delta F(y)}$ is a 'derivation', *i.e.* if it verifies the following properties

$$\frac{\delta}{\delta F(y)} (\lambda_1 A_1[F] + \lambda_2 A_2[F]) = \lambda_1 \frac{\delta A_1[F]}{\delta F(y)} + \lambda_2 \frac{\delta A_2[F]}{\delta F(y)} \tag{6}$$

$$\frac{\delta}{\delta F(y)} (A_1[F] A_2[F]) = A_1[F] \frac{\delta A_2[F]}{\delta F(y)} + A_2[F] \frac{\delta A_1[F]}{\delta F(y)} \tag{7}$$

then one checks by recurrence that $\frac{\delta (F(y_1)^k)}{\delta F(y_2)} = k F(y_1)^{k-1} \delta(y_2 - y_1)$, and thus the rule (3,4) is verified (at least for F expandable in series) since

$$\frac{\delta}{\delta F(y_2)} \int dy_1 \overbrace{\sum_k a_k(y_1) F(y_1)^k}^{\equiv \mathcal{L}(F(y_1), y_1)} = \int dy_1\, \delta(y_2 - y_1) \overbrace{\sum_k k a_k(y_1) F(y_1)^{k-1}}^{= \partial_F \mathcal{L}(F(y_1), y_1)} \tag{8}$$

Note that another good construction which implies (3,4) is

$$\boxed{\frac{\delta \mathcal{S}[F]}{\delta F(y)} = \lim_{\epsilon \to 0} \frac{\mathcal{S}[F + \epsilon\, \delta(. - y)] - \mathcal{S}[F]}{\epsilon}} \quad \text{where} \quad \mathcal{S}[F + f(. - y)] \equiv \int dx\, \mathcal{L}(F(x) + f(x - y), x) \tag{9}$$

(ii) What about derivatives?

Applying again rule (3,4)

$$\frac{\delta F'(y_1)}{\delta F(y_2)} = \frac{\delta}{\delta F(y_2)} \int dy\, \delta(y-y_1) F'(y) \overset{\text{(ipp)}}{=} -\frac{\delta}{\delta F(y_2)} \int dy\, \delta'(y-y_1) F(y) \tag{10}$$

one obtains (noting also that the distribution δ' is odd)

$$\boxed{\frac{\delta F'(y_1)}{\delta F(y_2)} = -\delta'(y_2-y_1) = \delta'(y_1-y_2)} \tag{11}$$

More generally, if the Lagrangian depends on the derivative of F in the action:

$$\mathcal{S}[F] = \int dy\, \mathcal{L}(F(y), F'(y), y) \tag{12}$$

taking (9) as a generic definition, one has

$$\begin{aligned}
\frac{\delta \mathcal{S}[F]}{\delta F(y)} &= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dx\, \Big[\mathcal{L}(F(x)+\epsilon\delta(x-y), F'(x)+\epsilon\delta'(x-y), x) - \mathcal{L}(F(y), F'(y), y)\Big] && (13)\\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dx\, \Big[\epsilon\delta(x-y)\partial_F \mathcal{L}(F(x), F'(x), x) + \epsilon\delta'(x-y)\partial_{F'}\mathcal{L}(F(x), F'(x), x)\Big] && (14)\\
&\overset{\text{(ipp)}}{=} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int dx\, \epsilon\delta(x-y)\Big[\partial_F \mathcal{L}(F(x), F'(x), x) - \frac{\partial}{\partial x}\Big\{\partial_{F'}\mathcal{L}(F(x), F'(x), x)\Big\}\Big] && (15)\\
&= \partial_F \mathcal{L}(F(y), F'(y), y) - \frac{\partial}{\partial y}\Big\{\partial_{F'}\mathcal{L}(F(y), F'(y), y)\Big\} && (16)
\end{aligned}$$

This implies (11) by writing $F'(y_1) = \int dy\, \delta(y-y_1)F'(y)$. Conversely, the relation (16) is also a consequence of (11), since (11) implies by recurrence

$$\frac{\delta(F(y_1)^k F'(y_1)^{k'})}{\delta F(y_2)} = kF(y_1)^{k-1}\delta(y_1-y_2)F'(y_1)^{k'} + k'F'(y_1)^{k'-1}\delta'(y_1-y_2)F(y_1)^k \tag{17}$$

which one can use as in (8) for $\mathcal{L}$ expanded in series as $\mathcal{L} = \sum_{k,k'} a_{kk'}(y)F(y)^k F'(y)^{k'}$. This shows that the generic definition (9) is equivalent to the computation rules (5,6,7,11).

More generically for higher order derivative dependence:

$$\begin{aligned}
\frac{\delta \mathcal{S}[F]}{\delta F(y)} = \partial_F \mathcal{L}(F(y), F'(y), \ldots, y) &- \frac{\partial}{\partial y}\Big\{\partial_{F'}\mathcal{L}(F(y), F'(y), \ldots, y)\Big\} \\
&+ \frac{\partial^2}{\partial y^2}\Big\{\partial_{F''}\mathcal{L}(F(y), F'(y), \ldots, y)\Big\} + \ldots \\
&+ (-1)^k \frac{\partial^k}{\partial y^k}\Big\{\partial_{F^{(k)}}\mathcal{L}(F(y), F'(y), \ldots, y)\Big\} + \ldots
\end{aligned} \tag{18}$$

(iii) Examples

The functional derivative allows to find a condition for an expression $\mathcal{L}(F(y), F'(y), y)$ to be an "total derivative":

$$\mathcal{L}(F(y), F'(y), y) \text{ is an total derivative} \tag{19}$$

$$\iff \exists f(F, F', y) : \mathcal{L}(F(y), F'(y), y) = \frac{\partial}{\partial y}[f(F(y), F'(y), y)] \tag{20}$$

$$\iff \forall A > 0, \int_{-A}^{A} dy\, \mathcal{L}(F(y), F'(y), y) = \Big[f(F(y), F'(y), y)\Big]_{-A}^{A}$$

$$\text{is independent of } F(y') \text{ for all } |y'| < A \tag{21}$$

$$\iff \forall A > 0, \forall |y'| < A, \frac{\delta}{\delta F(y')} \int_{-A}^{A} dy\, \mathcal{L}(F(y), F'(y), y) = 0 \tag{22}$$

$$\iff \frac{\delta}{\delta F(y')} \int dy\, \mathcal{L}(F(y), F'(y), y) = 0 \tag{23}$$

$$\iff \partial_F \mathcal{L}(F(y), F'(y), y) - \frac{\partial}{\partial y}\Big\{\partial_{F'} \mathcal{L}(F(y), F'(y), y)\Big\} = 0 \tag{24}$$

Some functional derivatives of the (functional) Gaussian in $F(y)$

$$G_0[F] = \exp\left[-\frac{1}{2}\int dy_1 dy_2\, F(y_1) C(y_2 - y_1) F(y_2)\right] \tag{25}$$

where $C(y)$ is an even function[1], write

$$\frac{\delta G[F]}{\delta F(y)} = \int dy_1 C(y_1 - y) F(y_1)\ , \qquad \frac{\delta^2 G[F]}{\delta F(y) \delta F(y')} = C(y' - y) \tag{26}$$

For the Gaussian in $F'(y)$ with $C(y) = \delta(y)$:

$$G_1[F, \lambda] = \exp\left[-\frac{\lambda}{2}\int dy\, F'(y)^2\right] \tag{27}$$

one has

$$\frac{\delta G_1[F, \lambda]}{\delta F(y)} = \lambda F''(y) G_1[F, \lambda] \tag{28}$$

This shows that, for $\lambda = 1$, $G_1[F, \lambda]$ is a steady solution to the Fokker-Planck equation

$$0 = \int dy \frac{\delta}{\delta F(y)} \Big\{ \underbrace{-F''(y) G[F] + \frac{\delta G[F]}{\delta F(y)}}_{=0 \text{ for } G[F] = G_1[F,1]} \Big\} \tag{29}$$

Note that if one computes the full right hand site for any λ, one finds

$$\int dy \frac{\delta}{\delta F(y)} \Big\{ (\lambda - 1) F''(y) G_1[F, \lambda] \Big\} = (\lambda - 1) \int dy\, [\delta''(0) + \lambda F''(y)^2] G_1[F, \lambda] \tag{30}$$

which involves a seemingly undefined term $(\lambda - 1) \int dy\, \delta''(0)$. However this term can be put without harm to 0 for $\lambda \to 1$ if one wants the following exchange of limits to hold

$$\lim_{\lambda \to 1} \int dy \frac{\delta}{\delta F(y)} \Big\{ -F''(y) + \frac{\delta G_1[F, \lambda]}{\delta F(y)} \Big\} = \int dy \frac{\delta}{\delta F(y)} \Big\{ \lim_{\lambda \to 1} \Big[-F''(y) + \frac{\delta G_1[F, \lambda]}{\delta F(y)} \Big] \Big\} \tag{31}$$

[1] It is always possible to write (25) with $C(y)$ an even function of y: if $C(y)$ is not even, take $\frac{1}{2}[C(y) + C(-y)]$.

Discrete variables

(i) Definitions

A continuum action $\mathcal{S}$ is the integral of a Lagrangian $\mathcal{L}$ which may not only depend on $F(y)$ but also on the derivatives $F'(y)$, $F''(y)$, ...: this non-strictly local dependence in $F(y)$ is inducing non-elementary functional derivatives as in (18). Consider now a set $\mathbf{F} = (F_i)$ of variables indexed by integers i. The discrete equivalent of this non-strictly local dependence is to have an action written as a sum of Lagrangians $\mathcal{L}_j$ depending on j not only through F_j but also through its neighbours, *e.g.* $F_{j\pm1}$. This dependence may always be written as

$$\mathcal{S}[\mathbf{F}] = \sum_j \mathcal{L}(F_j, \nabla_j^+ F, \nabla_j^- F, \Delta_j F, j) \tag{32}$$

where we have introduced the following discrete gradient and Laplacian operators

$$\nabla_i^+ F = F_{i+1} - F_i \qquad\qquad \nabla_i^- F = F_i - F_{i-1} \tag{33}$$

$$\Delta_i F = \nabla_i^+ \nabla_i^- F = (\nabla_i^+ - \nabla_i^-) F = F_{i+1} - 2F_i + F_{i-1} \tag{34}$$

The writing (32) is not unique (since one may for instance always replace $\Delta_j F$ by $\nabla_j^+ F - \nabla_j^- F$) but results do not depend on the choice of writing. One denotes for short by $\partial_F \mathcal{L}$ (resp. $\partial_+ \mathcal{L}$, $\partial_- \mathcal{L}$, $\partial_\Delta \mathcal{L}$, $\partial_i \mathcal{L}$) the derivative of the function $\mathcal{L}$ in (32) with respect to its 1st (resp. 2nd, 3rd, 4th, 5th) argument.

(ii) Derivation rules

Because of discreteness, the action (32) still depends explicitly on the individual F_i's. To compute $\partial \mathcal{S}[\mathbf{F}]/\partial F_i$ one thus only needs to identify the indices in the sum for which F_i appears. Noting for short $\mathcal{L}_j = \mathcal{L}(F_j, \nabla_j^+ F, \nabla_j^- F, \Delta_j F, j)$ (and the same for the derivatives $\partial_\pm \mathcal{L}$) and using the symbols (33,34) for $\mathcal{L}$, one thus has:

$$\frac{\partial \mathcal{S}[\mathbf{F}]}{\partial F_i} = \frac{\partial}{\partial F_i} \sum_j \mathcal{L}(\underbrace{F_j}_{\substack{\text{selects}\\ i=j}}, \underbrace{\nabla_j^+ F}_{\substack{\text{selects}\\ i=j+1\\ i=j}}, \underbrace{\nabla_j^- F}_{\substack{\text{selects}\\ i=j\\ i=j-1}}, \underbrace{\Delta_j F}_{\substack{\text{selects}\\ i=j+1\\ i=j\\ i=j-1}}, j) \tag{35}$$

and writing for the selected indices $j = i, i \pm 1$

$$\frac{\partial}{\partial F_i} \mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) = \partial_F \mathcal{L}_i - \partial_+ \mathcal{L}_i + \partial_- \mathcal{L}_i - 2\partial_\Delta \mathcal{L}_i \tag{36}$$

$$\frac{\partial}{\partial F_i} \mathcal{L}(F_{i+1}, \nabla_{i+1}^+ F, \nabla_{i+1}^- F, \Delta_{i+1} F, i+1) = -\partial_- \mathcal{L}_{i+1} + \partial_\Delta \mathcal{L}_{i+1} \tag{37}$$

$$\frac{\partial}{\partial F_i} \mathcal{L}(F_{i-1}, \nabla_{i-1}^+ F, \nabla_{i-1}^- F, \Delta_{i-1} F, i-1) = \partial_+ \mathcal{L}_{i-1} + \partial_\Delta \mathcal{L}_{i-1} \tag{38}$$

one finds, summing those lines

$$\boxed{\frac{\partial \mathcal{S}[\mathbf{F}]}{\partial F_i} = \partial_F \mathcal{L}_i - \nabla_i^- \partial_+ \mathcal{L} - \nabla_i^+ \partial_- \mathcal{L} + \Delta_i \partial_\Delta \mathcal{L}} \tag{39}$$

which is the discrete equivalent of the continuum functional derivative (18) with dependence up to the second derivative. Note in (39) that the two possible discretizations $\nabla_i^\pm F$ of the derivative $F'(y)$ yield a similar contribution to $\partial \mathcal{S}[\mathbf{F}]/\partial F_i$: to a dependence of $\mathcal{L}_i$ in $\nabla_i^+ F$ corresponds a discrete difference ∇_i^- in $\partial \mathcal{S}[\mathbf{F}]/\partial F_i$ (and conversely).

(ii) Necessary and sufficient condition for being a discrete difference

As in the continuum, one may find a condition for a function $\mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i)$ to be a discrete difference:

$$\mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) \text{ is a discrete difference} \tag{40}$$

$$\Longleftrightarrow \exists f_i \equiv f(F_{i-1}, F_i, F_{i+1}, i) : \mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) = \nabla_i^+ f \tag{41}$$

$$\Longleftrightarrow \forall A > 0, \sum_{|i|\le A} \mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) = f_{A+1} - f_{-A}$$

$$\text{is independent of } F_i \text{ for all } |i| < A \tag{42}$$

$$\Longleftrightarrow \forall A > 0, \forall |j| < A, \frac{\partial}{\partial F_j} \sum_{|i|\le A} \mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) = 0 \tag{43}$$

$$\Longleftrightarrow \frac{\partial}{\partial F_j} \sum_i \mathcal{L}(F_i, \nabla_i^+ F, \nabla_i^- F, \Delta_i F, i) = 0 \tag{44}$$

$$\Longleftrightarrow \partial_F \mathcal{L}_i - \nabla_i^- \partial_+ \mathcal{L} - \nabla_i^+ \partial_- \mathcal{L} + \Delta_i \partial_\Delta \mathcal{L} = 0 \tag{45}$$

This allows for instance to show that the only possibility for the following expression (arising from a discretized version of the KPZ Fokker-Planck equation)

$$\Big[a_1 (\nabla_i^+ F)^2 + a_2 \nabla_i^+ F\, \nabla_i^- F + a_3 (\nabla_i^- F)^2\Big] [\nabla_i^+ F - \nabla_i^- F] \tag{46}$$

to be a discrete difference is $a_1 = a_2 = a_3$ (in which case it is equal to $a_1((\nabla_i^+ F)^3 - (\nabla_i^- F)^3)$).

(iii) Discrete Gaussians

Let's consider the following discrete Gaussian distribution, analog to the continuum (27)

$$G_1[\mathbf{F}, \lambda] = \exp\Big[-\frac{\lambda}{2}\sum_i (F_{i+1} - F_i)^2\Big] = \exp\Big[-\frac{\lambda}{2}\sum_i (\nabla_i^+ F)^2\Big] \tag{47}$$

Using (39) one has

$$\frac{\partial G_1[\mathbf{F}, \lambda]}{\partial F_i} = \lambda \nabla_i^- \nabla_i^+ F\, G_1[\mathbf{F}, \lambda] = \lambda \Delta_i F\, G_1[\mathbf{F}, \lambda] \tag{48}$$

This shows that, for $\lambda = 1$, $G_1[\mathbf{F}, \lambda]$ is a steady solution to the discrete Fokker-Planck equation

$$0 = \sum_i \frac{\partial}{\partial F_i} \Big\{ \underbrace{-\Delta_i F G[\mathbf{F}] + \frac{\partial G[\mathbf{F}]}{\partial F_i}}_{=0 \text{ for } G[\mathbf{F}] = G_1[\mathbf{F},1]} \Big\} \tag{49}$$

Note that if one computes the full right hand site for any λ, one finds

$$\sum_i \frac{\partial}{\partial F_i}\Big\{(\lambda - 1)\Delta_i F\, G_1[\mathbf{F}, \lambda]\Big\} = (\lambda - 1)\sum_i \Big\{-2 + \lambda\, (\Delta_i F)^2\Big\} G_1[\mathbf{F}, \lambda] \tag{50}$$

(where we used $\frac{\Delta_i F}{\partial F_i} = -2$) which involves an infinite term $(\lambda - 1)\sum_i (-2)$, as in the continuum case (30). It is due to the fact that $\lim_{\lambda\to 1}$ and $\sum_i \frac{\partial}{\partial F_i}$ may no commute if $\sum_i \frac{\partial}{\partial F_i}$ does not exist before taking $\lim_{\lambda\to 1}$. However to enforce this commutation and write

$$\lim_{\lambda\to 1} \sum_i \frac{\partial}{\partial F_i}\Big\{ -\Delta_i F G_1[\mathbf{F}, \lambda] + \frac{\partial G[\mathbf{F}]}{\partial F_i}\Big\} = \sum_i \frac{\partial}{\partial F_i}\Big\{\lim_{\lambda\to 1}\Big[-\Delta_i F G_1[\mathbf{F}, \lambda] + \frac{\partial G[\mathbf{F}]}{\partial F_i}\Big]\Big\} = 0 \tag{51}$$

one may simply adopt the rule $\lim_{\lambda\to 1}\big[(\lambda - 1)\sum_i (-2)\big] = 0$.

Integration by parts

We assume again appropriate boundary conditions at infinity for border terms to vanish.

(o) One variable:

$$\int dF\, \mathcal{L}_1(F)\mathcal{L}_2'(F) = -\int dF\, \mathcal{L}_1'(F)\mathcal{L}_2(F) \tag{52}$$

(i) Discrete variables $\mathbf{F} = (F_i)$*:*

$$\int \prod_j dF_j \sum_i \mathcal{L}_1[\mathbf{F}]\frac{\partial \mathcal{L}_2[\mathbf{F}]}{\partial F_i} \;=\; -\int \prod_j dF_j \sum_i \frac{\partial \mathcal{L}_1[\mathbf{F}]}{\partial F_i}\mathcal{L}_2[\mathbf{F}] \tag{53}$$

(ii) Functional integrals:

$$\int \mathcal{D}F \int dy\, \mathcal{L}_1[F]\frac{\delta \mathcal{L}_2[F]}{\delta F(y)} \;=\; -\int \mathcal{D}F \int dy\, \frac{\delta \mathcal{L}_1[F]}{\delta F(y)}\mathcal{L}_2[F] \tag{54}$$

(iii) Application to the (functional) Fokker-Planck equation:

To the Langevin equation

$$\partial_t F(t,y) = \mathcal{G}[F,y] + V(t,y) \tag{55}$$

where $V(t,y)$ is a Gaussian noise with with correlations $\langle V(t,y)V(t',y')\rangle = D\delta(t'-t)R_\xi(y'-y)$ corresponds the Fokker-Planck equation

$$\partial_t P[F,t] = \int dy\, \frac{\delta}{\delta F(y)}\left[-\mathcal{G}[F,y]P[F,t] + \frac{D}{2}\int dy' R_\xi(y'-y)\frac{\delta P[F,t]}{\delta F(y')}\right] \tag{56}$$

The time-derivative of the average of an observable $\mathcal{O}(F(y))$ at time t and fixed y_1 writes

$$\begin{aligned}
\partial_t \Big\langle \mathcal{O}(F(t,y_1))\Big\rangle &= \int \mathcal{D}F\, \mathcal{O}(F(y_1))\, \partial_t P[F,t] &(57)\\
&\overset{(56)}{=} \int \mathcal{D}F \int dy\, \mathcal{O}(F(y_1))\, \frac{\delta}{\delta F(y)}\left[-\mathcal{G}[F,y]P[F,t] + \frac{D}{2}\int dy' R_\xi(y-y')\frac{\delta P[F,t]}{\delta F(y')}\right] &(58)\\
&\overset{(54)}{=} \int \mathcal{D}F \int dy\, \underbrace{\frac{\delta \mathcal{O}(F(y_1))}{\delta F(y)}}_{=\delta(y-y_1)\partial_F \mathcal{O}(F(y_1))}\left[\mathcal{G}[F,y]P[F,t] - \frac{D}{2}\int dy' R_\xi(y-y')\frac{\delta P[F,t]}{\delta F(y')}\right] &(59)\\
&= \int \mathcal{D}F\, \partial_F \mathcal{O}(F(y_1))\left[\mathcal{G}[F,y_1]P[F,t] - \frac{D}{2}\int dy' R_\xi(y_1-y')\frac{\delta P[F,t]}{\delta F(y')}\right] &(60)\\
&\overset{(54)}{=} \langle \mathcal{G}[F,y_1]\partial_F \mathcal{O}(F(t,y_1))\rangle + \frac{D}{2}\int \mathcal{D}F \int dy' \partial_F^2 \mathcal{O}(F(y_1))\delta(y'-y_1)R_\xi(y_1-y')P[F,t] &(61)\\
&= \langle \mathcal{G}[F,y_1]\partial_F \mathcal{O}(F(t,y_1))\rangle + \frac{D}{2}\int \mathcal{D}F\, \partial_F^2 \mathcal{O}(F(y_1))\, R_\xi(0)\, P[F,t] &(62)
\end{aligned}$$

and finally

$$\boxed{\partial_t \Big\langle \mathcal{O}(F(t,y))\Big\rangle = \Big\langle \mathcal{G}[F,y]\partial_F \mathcal{O}(F(t,y))\Big\rangle + \frac{D}{2}R_\xi(0)\Big\langle \partial_F^2 \mathcal{O}(F(t,y))\Big\rangle} \tag{63}$$

For $\mathcal{O}(F) = F$, one finds

$$\partial_t \langle F(t,y)\rangle = \langle \mathcal{G}[F,y]\rangle \tag{64}$$

which is somehow obvious from the Langevin equation (55) (since V is centered), but instructive to obtain in the context of the Fokker-Planck equation.

For $\mathcal{O}(F) = F^2$ one finds

$$\partial_t \langle F(t,y)^2 \rangle = 2\langle F(t,y)\mathcal{G}[F,y]\rangle + DR_\xi(0) \tag{65}$$

which is singular for $\xi \to 0$ and not obvious to obtain from the Langevin equation (55), since (55) would formally imply

$$\begin{aligned}\partial_t \langle F(t,y)^2 \rangle &= 2\langle F(t,y)\partial_t F(t,y)\rangle &(66)\\ &= 2\langle F(t,y)\mathcal{G}[F,y]\rangle + 2\langle F(t,y)V(t,y)\rangle &(67)\end{aligned}$$

The last term is difficult to compute and (may) correspond(s) to the last one of (65).

Another example is provided by the computation of the average of multiple point time derivatives:

$$\partial_t \langle F(y_1)F(y_2)\rangle = \int \mathcal{D}F \int dy\, \underbrace{\frac{\delta(F(y_1)F(y_2))}{\delta F(y)}}_{=\delta(y-y_1)F(y_2)+\delta(y-y_2)F(y_1)} \left[\mathcal{G}[F,y]P[F,t] - \frac{D}{2}\int dy' R_\xi(y-y')\frac{\delta P[F,t]}{\delta F(y')}\right] \tag{68}$$

$$\begin{aligned}&= \int \mathcal{D}F \left\{ F(y_1)\left[\mathcal{G}[F,y_2]P[F,t] - \frac{D}{2}\int dy' R_\xi(y_2-y')\frac{\delta P[F,t]}{\delta F(y')}\right]\right.\\ &\qquad \left. + F(y_2)\left[\mathcal{G}[F,y_1]P[F,t] - \frac{D}{2}\int dy' R_\xi(y_1-y')\frac{\delta P[F,t]}{\delta F(y')}\right]\right\} &(69)\\ &= \Big\langle \mathcal{G}[F,y_1]F(t,y_2) + \mathcal{G}[F,y_2]F(t,y_1)\Big\rangle\\ &\qquad + \frac{D}{2}\int \mathcal{D}F \int dy' \Big\{R_\xi(y_2-y')\delta(y_1-y') + R_\xi(y_1-y')\delta(y_2-y')\Big\}P[F,t] &(70)\end{aligned}$$

and finally

$$\boxed{\partial_t \Big\langle F(t,y_1)F(t,y_2)\Big\rangle = \Big\langle \mathcal{G}[F,y_1]F(t,y_2) + \mathcal{G}[F,y_2]F(t,y_1)\Big\rangle + DR_\xi(y_2-y_1)} \tag{71}$$

which yields back (65) for $y_1 = y_2$. More generally, one has (noting $\partial_1\mathcal{O}$ (resp. $\partial_2\mathcal{O}$) the derivative of $\mathcal{O}$ w.r.t its first (resp. second) argument):

$$\begin{aligned}\partial_t \Big\langle \mathcal{O}(F(t,y_1),F(t,y_2))\Big\rangle =& \Big\langle \mathcal{G}[F,y_1]\partial_1\mathcal{O}(F(t,y_1),F(t,y_2)) + \mathcal{G}[F,y_2]\partial_2\mathcal{O}(F(t,y_1),F(t,y_2))\Big\rangle\\ &+ \frac{D}{2}\Big[R_\xi(0)\partial_{11} + 2R_\xi(y_2-y_1)\partial_{12} + R_\xi(0)\partial_{22}\Big]\mathcal{O}(F(t,y_1),F(t,y_2))\end{aligned} \tag{72}$$