Letters

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Letter

Building a path-integral calculus: a covariant discretization approach

Leticia F Cugliandolo[1], Vivien Lecomte[2] and Frédéric van Wijland[3]

[1] Laboratoire de Physique Théorique et Hautes Énergies, Sorbonne Université, UMR 7589 CNRS, 4 Place Jussieu Tour 13 5ème étage, 75252 Paris Cedex 05, France
[2] Laboratoire Interdisciplinaire de Physique, Université Grenoble Alpes, CNRS, LIPhy, F-38000 Grenoble, France
[3] Laboratoire Matière et Systèmes Complexes, Université Paris Diderot, UMR 7057 CNRS, 10 rue Alice Domon et Léonie Duquet, 75205 Paris Cedex 13, France

E-mail: leticia@lpthe.jussieu.fr, vivien.lecomte@univ-grenoble-alpes.fr and frederic.van-wijland@univ-paris-diderot.fr

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Abstract

Path integrals are a central tool when it comes to describing quantum or thermal fluctuations of particles or fields. Their success dates back to Feynman who showed how to use them within the framework of quantum mechanics. Since then, path integrals have pervaded all areas of physics where fluctuation effects, quantum and/or thermal, are of paramount importance. Their appeal is based on the fact that one converts a problem formulated in terms of operators into one of sampling classical paths with a given weight. Path integrals are the mirror image of our conventional Riemann integrals, with functions replacing the real numbers one usually sums over. However, unlike conventional integrals, path integration suffers a serious drawback: in general, one cannot make non-linear changes of variables without committing an error of some sort. Thus, no path-integral based calculus is possible. Here we identify which are the deep mathematical reasons causing this important caveat, and we come up with cures for systems described by one degree of freedom. Our main result is a construction of path integration free of this longstanding problem, through a direct time-discretization procedure.

Keywords: path integrals, multiplicative Langevin equations, functional calculus, diffusion processes

(Some figures may appear in colour only in the online journal)

1. Introduction

Though the notion of path integration can be traced back to Wiener [1, 2], it is fair to credit Feynman [3] for making path integrals one of the daily tools of theoretical physics. The idea is to express the transition amplitude of a particle between two states as an integral over all possible trajectories between these states with an appropriate weight for each of them. After such a formulation of quantum mechanics was proposed, path integrals turned out to provide a set of methods that are now ubiquitous in physics (see [4–6] for reviews) and they have become the language of choice for quantum field theory. But path integrals reach out well beyond quantum physics and they are also a versatile instrument to study stochastic processes. Beyond Wiener's original formulation of Brownian motion, Onsager and Machlup [7, 8], followed by Janssen [9, 10], and De Dominicis [11, 12] (based on the operator formulation of Martin *et al* [13]), have contributed to establish path integrals as a useful tool, on equal footing with the Fokker–Planck and Langevin equations. The gist of the mathematical difficulty is to manipulate signals that are nowhere differentiable. Interestingly, mathematicians have mostly stayed a safe distance away from path integrals. Indeed, it has been known for many years that path integrals cannot be manipulated without extra caution in a vast category of problems. These problems, in the stochastic language, involve the notion of multiplicative noise (that we describe in detail below), and their counterpart in the quantum world has to do with quantization on curved spaces [14].

The late seventies witnessed an important step toward the understanding of the subtleties of path integrals: the authors of [15–20] found how to formulate path integrals in terms of smooth (differentiable) functions. By construction, their formulation does not offer a direct interpretation in terms of the weight of the physical trajectories, which are non-differentiable. The goal of this article is to come up with the missing link: we construct path integrals for non-differentiable stochastic and/or quantum trajectories, free of any mathematical hitch, by a direct time-discretization procedure which endows them with a well-defined mathematical meaning, consistent with differential calculus.

2. Result and outline

Consider a system described by a single degree of freedom $x(t)$ with noisy dynamics (i.e. subjected to a random force). We give an unambiguous definition of the probability density $\mathbb{P}$ of a path $[x(t)]_{0\leqslant t\leqslant t_\mathrm{f}}$ in a form that is *covariant* under any change of variables $u(t) = U(x(t))$. Namely, denoting by x_k and u_k the sequences of values that the paths $x(t)$ and $u(t)$ take at discrete times indexed by an integer k, the probability density $\mathbb{P}$ of such sequences satisfies

$$\prod_{k=0}^{N} \mathrm{d}x_k \, \mathbb{P}_X[\{x_\ell\}] = \prod_{k=0}^{N} \mathrm{d}u_k \, \mathbb{P}_U[\{u_\ell\}], \tag{1}$$

with $\mathbb{P}_X$ and $\mathbb{P}_U$ taking *the same functional form* for the processes $\{x_\ell\}$ and $\{u_\ell = U(x_\ell)\}$. In these expressions, U is an arbitrary invertible differentiable function and N is the number of time steps in which the time window $[0, t_\mathrm{f}]$ is divided. The precise definitions of all the entities involved in the relation (1) will be given in the central part of this paper (section 4).

The continuous-time limit of $\mathbb{P}[x]$ reads $\mathcal{N}[x]\mathrm{e}^{-S[x]}$ where both the 'action' $S[x]$ and the 'normalization factor' $\mathcal{N}[x]$ are covariant: in the Lagrangian writing $S[x] = \int_0^{t_\mathrm{f}} \mathrm{d}t\, \mathcal{L}(x, \dot{x})$, switching between $x(t)$ and $u(t)$ merely amounts to applying the chain rule $\dot{u}(t) = \dot{x}(t)\, U'(x(t))$. We emphasize that, from our theoretical physicist's point of view, $\mathbb{P}[x]$ acquires the meaning of the probability of a path only when a discretized version is given and such a discretization issue is not a mathematical detail: continuous-time writings of $\mathcal{N}[x]$ and $S[x]$ do not allow one to identify without ambiguity the probability of a path[4]. The discretization scheme that we present in this work is compatible with the covariance relation (1) and solves the long-standing problem of building a well-defined path probability that is consistent with differential calculus.

In what follows, we construct our path integral by carefully manipulating non-differentiable trajectories, directly from a Langevin equation. The latter suffers from ambiguities that only a discretized formulation can waive, and we thus begin in section 3 with a review of discretization issues in Langevin equations. With this settled, we present in section 4 the main outcome of our paper, a path probability (that includes a carefully defined normalization factor) that allows one to use the standard rule of calculus inside the action when changing variables, even in the time-discrete formulation (1) and for non-differentiable trajectories. Constructing the actual time-discrete path probability requires to focus on hitherto overlooked contributions in slicing up time-evolution, but also to resort to a new adaptive slicing of time. It amounts to identifying the correct discretization of the integral $S[x] = \int_0^{t_f} \mathrm{d}t\, \mathcal{L}(x, \dot{x})$, an issue that goes well beyond the usual Itō-Stratonovich dilemma, and thus enforces us to implement a generalization of the standard stochastic integral. In section 5 we compare our result and our construction to other path-integral formulations. In section 6, we then show how to transpose our construction to the so-called Martin–Siggia–Rose–Janssen–De Dominicis [9–13] (MSRJD) path-integral representation of the path probability, that provides the Hamiltonian counterpart of the former Lagrangian formulation [the action $S[\hat{x}, x]$ now depending on a 'response variable' $\hat{x}(t)$ conjugate to $x(t)$]. We finally provide in section 7 our conclusion and outlook.

3. Stochastic processes

For concreteness, we focus on the problem of a point-like particle moving in a one dimensional space.

3.1. Langevin's Langevin equation

Langevin introduced the celebrated equation that goes under his name to describe Brownian motion [24]. His idea was to start from Newton's equation $m\dot{v}(t) = F(t)$ for the motion of the large particle with mass m and velocity v, and to mimic the effect of its contact with the embedding liquid through a phenomenological force $F(t)$ made of two terms: a dissipative contribution, $-\gamma v(t)$, and a time-dependent random one, $\eta(t)$. With this simple choice for the former and adopting adequate statistical properties for the latter, he represented the observed erratic motion of the particle, and understood the behavior of varied experimentally averaged observables, constructed in his formalism as averages (denoted $\langle \cdot \rangle$) over the noise. Importantly enough, he assumed that the random force was Gaussian distributed at each instant, had zero mean, $\langle \eta(t) \rangle = 0$, and was Dirac-delta correlated in time, $\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$, assuming a strong separation of time-scales between the one of the motion of the Brownian particle and the ones typical of the motion of the constituents of the 'bath'. Such a 'thermal noise' $\eta(t)$ is

[4] We underline here an important cultural difference with a mathematician's viewpoint which would consist in defining a path-integral action directly in continuous time, following Wiener [1, 2] (and others [21–23] for multiplicative processes). Our point of view is different: we prefer to keep an underlying time discretization with infinitesimal time step Δt which allows us to control the rest in powers of Δt when manipulating the action, evaluated on non-differentiable paths. From a mathematician's viewpoint, we are interested in the probability density of the events $\{x(t_k) = x_k\}$.

termed *Gaussian white noise*. Denoting by k_{B} the Boltzmann constant and by T the ambient temperature, the parameter D is fixed to $\gamma k_{\mathrm{B}} T$ in order to ensure kinetic energy equipartition. In the so-called overdamped limit one studies time-scales that are much longer than m/γ, neglecting inertia compared to the effect of other forces, and focuses on the particles's position $x(t)$ that is ruled by $\dot{x}(t) = f(x(t)) + \eta(t)$. In this notation the friction coefficient γ was absorbed in a redefinition of time, and a term $f(x(t))$, proportional to an external force, was added to describe more general physical situations.

3.2. Reductionism: other Langevin equations

Stochastic equations of the Langevin kind have later been derived for the dynamics of other degrees of freedom than the position, or of fluctuating order parameters in (even originally quantum) systems, after a *model reduction* that amounts to integrating over a large number of degrees of freedom in an interacting system, keeping only a few representative ones. The range of applicability of Langevin equations therefore became much wider than originally expected [25, 26]. A large separation of time-scales is also usually advocated to claim that Gaussian white noise is a reasonable choice and, furthermore, the overdamped limit is also often justified.

3.3. Multiplicative noise

In many cases of practical interest the noise is not additive as in Langevin's original proposal but appears multiplied by a function of the variable of interest,

$$\dot{x}(t) = f(x(t)) + g(x(t))\eta(t), \tag{2}$$

still with $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. Such a multiplicative noise is involved in a flurry of physical problems ranging from soft matter (e.g. diffusion in microfluidic devices [27]), to condensed matter (e.g. super paramagnets [28, 29]) or even inflational cosmology [30, 31]. It appears in other areas of science in which Langevin equations are present (e.g. Black–Scholes equation for option pricing [32]). Quantization on curved spaces (e.g. a particle on a sphere [33, 34] or more generic manifolds [35–38]) pertains to the same mathematical class of problems, even though their physical motivation has a different origin. Connections between thermal and quantum noises were noted by Nelson [39], and it is therefore no surprise that our discussion addresses both class of problems.

3.4. Discretization

Langevin defined his equation and performed calculations in a continuous-time setting. However, an overdamped multiplicative Langevin equation such as (2) acquires a well-defined meaning only if a discretization scheme is chosen. We adopt here the physicist's description where a time-discrete version of (2) is made explicit. Controlling the zero time-step limit in a careful way is crucial when dealing with stochastic equations because $x(t)$ is not a differentiable function. To address this issue with the appropriate rigor, mathematicians have developed the field of stochastic calculus (see for instance [40, 41] for reviews); thus, they often use the continuous-time Wiener measure as a reference to define other structures of interest, but we do not follow this approach here because our interest goes to explicit trajectory weights.

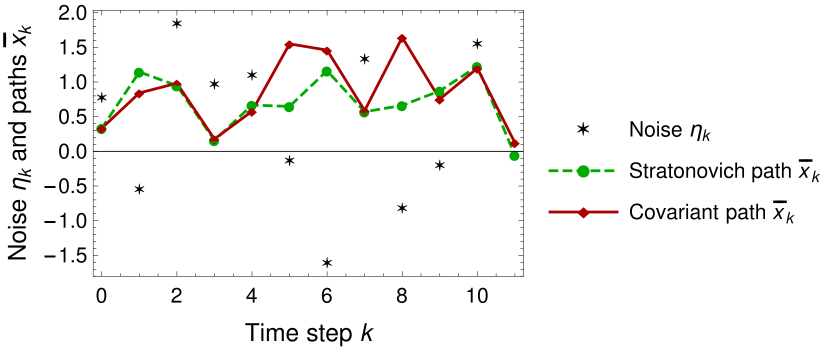

Figure 1. Comparison between Langevin paths discretized à la Stratonovich (circles, dashed line) and with the covariant rule (diamonds, full line), according to equations (5) and (9), respectively. The noise is represented with stars. The Langevin dynamics is defined by $f(x) = 1 + x$, $g(x) = 4x^4$, $D = 1$, discretized with $\Delta t = 1/4$.

The time interval $[0, t_\mathrm{f}]$ is divided into N steps of equal duration Δt, in such a way that $t_k = k\Delta t$, with $k = 0, \ldots, N$ and $N\Delta t = t_N = t_\mathrm{f}$. The instantaneous noise $\eta_k = \eta(t_k)$, is drawn from the joint probability distribution function (pdf)

$$\mathbb{P}[\{\eta_k\}] = \prod_{0 \leqslant k < N} \sqrt{\frac{\Delta t}{4\pi D}}\, \mathrm{e}^{-\frac{\Delta t}{4D}\eta_k^2}. \tag{3}$$

A set of noises drawn from this pdf are shown in figure 1 with stars. The measure over which functions of the noise are integrated over is $\mathcal{D}\eta \equiv \prod_{k=0}^{N-1} \mathrm{d}\eta_k$. This pdf implies $\langle \eta_k \rangle = 0$ and $\langle \eta_k \eta_{k'} \rangle = 2D\delta_{kk'}/\Delta t$, making explicit that $\eta_k = O(\Delta t^{-1/2})$ at each time step. To define the Langevin equation (2), one now specifies the time-discrete evolution for the $x_k \equiv x(t_k)$'s (with $0 \leqslant k \leqslant N$). First, the time derivative $\dot{x}(t)$ evaluated at t_k represents the ratio $\Delta x/\Delta t$ between the two forward increments, $\Delta x \equiv x_{k+1} - x_k$ and $\Delta t = t_{k+1} - t_k$. Second, to specify how to evaluate $x(t)$ on the right-hand side (r.h.s.) of (2), we denote by $\bar{x}_k$ the arguments of the functions f and g in the time-discrete evolution. In conventional stochastic calculus, $\bar{x}_k$ is given by a linear combination of x_k and x_{k+1}. A dependence on the sole pre-point $\bar{x}_k = x_k$ is chosen in the Itō scheme and, instead, the mid-point dependence $\bar{x}_k = (x_k + x_{k+1})/2$ is taken in the Stratonovich one. Each form has its advantages and drawbacks. Within the Stratonovich convention, in the continuous-time limit, one can manipulate $x(t)$ as if it were differentiable but x_{k+1} appears in implicit form in the discrete equation at time t_k and this is not convenient for numerical integration. Instead, the Itō scheme yields a recursion particularly suited to the computer generation of an individual trajectory. However, in contrast to equation (2) understood with the Stratonovich rule, one cannot manipulate $x(t)$ as if it were differentiable. This problem was addressed by mathematicians who modified the rules of calculus to be able to work with $x(t)$ in the continuous-time limit. This is the celebrated Itō's lemma [42].

The continuous-time equation (2) is thus understood as a short-hand writing which acquires a well-defined meaning *only* through a limiting procedure $\Delta t \to 0$ which starts from a discrete-time evolution in which a prescription (or 'discretization scheme') for $\bar{x}_k$ is given. We focus on the Stratonovich choice henceforth, with the aim of building a path-integral formalism in which the standard rules of calculus could also be used. The time-discrete evolution is therefore given by

$$\frac{\Delta x}{\Delta t} \stackrel{\mathrm{S}}{=} f(\bar{x}_k) + g(\bar{x}_k)\eta_k \qquad (0 \leqslant k < N), \tag{4}$$

$$\bar{x}_k = \frac{x_k + x_{k+1}}{2} = x_k + \frac{1}{2}\Delta x, \tag{5}$$

where $\stackrel{\mathrm{S}}{=}$ indicates that $\bar{x}_k$ is Stratonovich-discretized. This implies that the typical Δx is of order $\sqrt{\Delta t}$, and not Δt, reflecting the well-known fact that a Brownian motion is nowhere differentiable. Each choice of $\{\eta_k\}_{0\leqslant k<N}$ drawn from the noise pdf (3) yields a trajectory $\{x_k\}_{0\leqslant k\leqslant N}$, with x_0 drawn from a distribution P_i. A sketch of such a trajectory is shown in figure 1 with circles. The probability density (or 'path probability') of such trajectories, $\mathbb{P}_X[\{x_k\}]$, will be the object of our study.

3.5. Rules of calculus and covariance of the Langevin equation

Consider a change of variables $u(t) = U(x(t))$ of the process $x(t)$, where U is a differentiable and invertible function. Natural questions are: is it valid to use the chain rule to compute $\dot{u}(t)$? What is the Langevin equation governing the process $u(t)$? In discrete time, defining $u_k \equiv u(t_k) = U(x_k)$, one expresses $\Delta u \equiv u_{k+1} - u_k = U(x_k + \Delta x) - U(x_k)$ using a Taylor expansion in powers of the increment Δx,

$$\Delta u = U'(x_k)\Delta x + \frac{1}{2}U''(x_k)\Delta x^2 + O(\Delta x^3) \tag{6}$$

that, using $\bar{x}_k \stackrel{\mathrm{S}}{=} x_k + \frac{1}{2}\Delta x$ and $\Delta x = O(\Delta t^{1/2})$, becomes

$$\frac{\Delta u}{\Delta t} \stackrel{\mathrm{S}}{=} U'(\bar{x}_k)\frac{\Delta x}{\Delta t} + O(\Delta t^{1/2}). \tag{7}$$

In the continuous-time limit, the terms of order $\Delta t^{1/2}$ and higher in equation (7) are negligible, and one recovers the usual chain rule, $\dot{u}(t) \stackrel{\mathrm{S}}{=} U'(x(t))\dot{x}(t)$, within the Stratonovich scheme.

To determine the evolution equation verified by $u(t)$ in the Stratonovich scheme, one defines $\bar{u}_k = (u_k + u_{k+1})/2$. Inserting (4) into (7)[5], the time-discrete equation follows

$$\frac{\Delta u}{\Delta t} = F(\bar{u}_k) + G(\bar{u}_k)\eta_k + O(\Delta t^{1/2}) \tag{8}$$

where $F(u)$ and $G(u)$ are the force and the noise amplitude of the Langevin equation verified by $u(t)$, defined as $F(U(x)) = U'(x)f(x)$ and $G(U(x)) = U'(x)g(x)$.

Consistently with the chain rule, at leading order in Δt and using the inverse function that leads from $u(t)$ to $x(t)$, one recovers equation (4) for the original process $x(t)$ from equation (8), thus proving that a Stratonovich-discretized Langevin equation is covariant. In short, with the Stratonovich discretization, the standard chain rule of differential calculus can be used without caution most of the time, even though none of the manipulated objects is actually differentiable! (These properties are generalized to other *linear* discretization schemes $\bar{x}_k = x_k + \alpha\Delta x$, including the celebrated Itō one, once the rules of calculus are modified appropriately [25, 26, 43].)

The subleading terms of order $\Delta t^{1/2}$ in equations (7) and (8) show that the chain rule or the Langevin equation for $u(t)$ are not exact at finite Δt, but become valid only in the continuous-time limit. Computing such terms explicitly improves, for instance, the precision of numerical algorithms (inevitably defined in discrete time, see e.g. [44–49]). More importantly for

[5] And using $h(\bar{x}_k) \stackrel{\mathrm{s}}{=} h(U^{-1}(\bar{u}_k)) + O(\Delta t)$ valid for any function h.

our purposes, we will show that these subleading terms are responsible for the breakdown of covariance in the standard path-integral formalism. This raises a natural question that we address in the following section: whether there exists a discretization scheme for which the Langevin equations be exactly covariant, that is up to an arbitrary order in Δt.

3.6. Improved covariant discretization

A g-dependent discretization scheme of the form

$$\bar{x}_k \stackrel{\beta_g}{=} x_k + \frac{1}{2}\Delta x + \beta_g(x_k)\Delta x^2, \tag{9}$$

$$\beta_g(x) = \frac{1}{24}\frac{g''(x)}{g'(x)} - \frac{1}{12}\frac{g'(x)}{g(x)}, \tag{10}$$

yields an evolution equation (8) for $u(t)$ valid up to order Δt, namely one more order in $\Delta t^{1/2}$ than the Stratonovich one. The ensemble of points $\{x_\ell\}$ generated by one such Langevin equation are shown with diamonds in figure 1. Such a scheme, that we call *covariant discretization* (or for short β_g-discretization), serves as a starting point for our construction of the path integral, where the argument of every function in the action will be understood as discretized according to equation (9). As described in appendix A.1, a full series in powers of Δx can be added to equation (9) in order to yield a chain rule (8) that is exact to all orders in Δt (see the expansions of equations (A.2) or (A.5)). Yet, as we show later, the sole additional contribution $\beta_g(x)\Delta x^2$ in equation (9) is sufficient to immunize path integrals against the problems caused by nonlinear manipulations.

When used in the discrete Langevin equation (4), the covariant discretization (9) and (10) yields the same equation as the Stratonovich one in the $\Delta t \to 0$ limit: these two schemes are equivalent. An essential aspect of our construction is that such an equivalence becomes wrong in the path-integral action: as we will show, the covariant and the Stratonovich schemes are not equivalent discretizations when used in the Lagrangian.

Finally, note that for the covariant discretization to be well defined, we assume that the dynamics ensures that $x(t)$ stays in an interval of the real line where $g(x) > 0$ and $g''(x)/g'(x)$ remains finite.

4. Probability distribution function of a trajectory

We now focus on the construction of the path probability $\mathbb{P}_X[\{x_\ell\}]$. Such an expression is handy since with it one can directly compute the average of any observable of interest, $\mathcal{F}[x]$, as the path-integral $\langle \mathcal{F}[x] \rangle = \int \mathcal{D}x\, \mathcal{F}[x]\, \mathbb{P}_X[x]\, P_{\rm i}(x(0))$ interpreted in the Feynman sense [3]: a sum over all possible trajectories in discrete time with the measure defined as $\mathcal{D}x = \prod_{k=0}^{N} \mathrm{d}x_k$. The initial condition is sampled by $P_{\rm i}$. We will compare in section 5 our expression for the path probability to the many existing results in the literature.

4.1. Propagator

The path probability of a trajectory is inferred from the infinitesimal propagator $\mathbb{P}_X(x_1|x_0) \equiv \mathbb{P}(x_1, \Delta t|x_0, 0)$ for the first time step, defined as the conditional probability that $x(\Delta t) = x_1$ at time $t_1 = \Delta t$, given $x(0) = x_0$ at time $t_0 = 0$. Indeed, the full trajectory pdf reads

$$\mathbb{P}_X[\{x_\ell\}] = \prod_{0\leqslant k<N} \mathbb{P}(x_{k+1},t_{k+1}|x_k,t_k) \equiv \mathcal{N}_X[\{x_\ell\}]\,\mathrm{e}^{-S_X[\{x_\ell\}]}. \tag{11}$$

In the above formula we used a standard representation in which the path probability is written as the product of the exponential of an action $S[x]$ and a normalization factor $\mathcal{N}[x]$. Clearly, this separation is not unique as factors can be exponentiated in the action or vice versa. We adopt the convenient choice [20, 50]

$$\mathcal{N}_X[\{x_\ell\}] \equiv \prod_{0\leqslant k<N} \frac{1}{\sqrt{4\pi D\Delta t}}\,\frac{1}{|g(x_{k+1})|} \tag{12}$$

of discretizing at the endpoint, that is different from another standard convention in which the prefactor is discretized at $\bar{x}_k$ [51–54]. The reason for adopting (12) instead of the latter is that when changing paths from $\{x_\ell\}$ to $\{u_\ell\}$, the corresponding Jacobians and the conversion of the prefactor bring out factors $|U'(x_{k+1})|$ and $1/|U'(x_{k+1})|$ $(0\leqslant k<N)$ that cancel one by one, including at time boundaries. Another choice would lead to a normalization prefactor that is not covariant, implying that upon a change of variables extra terms coming from the prefactor would impact the action (see for instance [55]). The choice (12) allows one to focus on the sole transformation properties of the action in the exponential. The full form (11) is inferred from the elementary propagator for the first time step that we write as

$$\mathbb{P}_X(x_1|x_0) = \frac{1}{\sqrt{4\pi D\Delta t}\,|g(x_1)|}\,\mathrm{e}^{-\delta S_X^{\Delta t}}. \tag{13}$$

4.2. Stratonovich action

Following well-known routes [9, 51–60], one finds that, in the Stratonovich scheme, the elementary contribution $\delta S_X^{\Delta t}$ to the action for the first time step between x_0 and $x_0+\Delta x$ reads

$$\begin{aligned}\delta S_X^{\Delta t} \stackrel{\mathrm{S}}{=} \frac{1}{2}\frac{\Delta t}{2D}\Big[\frac{\frac{\Delta x}{\Delta t}-f(\bar{x}_0)}{g(\bar{x}_0)}\Big]^2 + \frac{\Delta t}{2}\Big[f'(\bar{x}_0) - \frac{f(\bar{x}_0)g'(\bar{x}_0)}{g(\bar{x}_0)}\Big] \\ + \frac{D}{4}\big[2g'(\bar{x}_0)^2 - g(\bar{x}_0)g''(\bar{x}_0)\big]\Delta t,\end{aligned} \tag{14}$$

that in the continuous-time writing yields the action

$$\begin{aligned}S_X^{\mathrm{S}}[x] \stackrel{\mathrm{S}}{=} \int_0^{t_{\mathrm{f}}} \mathrm{d}t\,\Big\{\frac{1}{4D}\Big[\frac{\dot{x}-f(x)}{g(x)}\Big]^2 + \frac{1}{2}f'(x) - \frac{f(x)g'(x)}{2g(x)} \\ + \frac{D}{4}\big[2g'(x)^2 - g(x)g''(x)\big]\Big\}.\end{aligned} \tag{15}$$

The reader can easily verify that this continuous-time action is not covariant. By this we mean that under a change of variables $x\mapsto U(x)$, and using the chain rule $\dot{u} = \dot{x}\,U'(x)$, one does not find the correct action S_U^{S}, that has the same form as S_X^{S} with the replacements $f\mapsto F$ and $g\mapsto G$ (and similarly if one tries to reconstitute S_X^{S} from S_U^{S}). Such problems were noted in the early developments of path integrals (see e.g. [37, 61, 62]). The reason, originally identified in [63], is actually simple: going for instance from u to x, the dominant term (of order Δt^0) in $\delta S_U^{\Delta t}$ is $\frac{\Delta t}{4DG(\bar{u}_0)^2}[\frac{\Delta u}{\Delta t}]^2$, see equation (14). Changing variables, one uses the Stratonovich-discretized chain rule (7). The dominant term in equation (7) yields the expected dominant term $\frac{\Delta t}{4Dg(\bar{x}_0)^2}[\frac{\Delta x}{\Delta t}]^2$ in $\delta S_X^{\Delta t}$, but the rest in equation (7) yields a double-product contribution

$\frac{\Delta t}{2Dg(\bar{x}_0)^2}\frac{\Delta x}{\Delta t}\times O(\Delta t^{1/2})$ which is of order Δt and thus *cannot be neglected*. The conclusion is simple: in the Stratonovich scheme, using the continuous-time chain rule $\dot{u}=\dot{x}\,U'(x)$ in the action yields a wrong result because the rest $O(\Delta t^{1/2})$ in equation (7) that could be neglected at the Langevin level (in equation (8) for instance) cannot be neglected in the action. While the Stratonovich discretization (5) was sufficient to render the Langevin equation (4) covariant, it fails to play the same role at the path-integral level. Changing variables is still possible in (15) but this involves highly intricate rules (see for instance [55]). At this stage, we recall the lesson of Edwards and Gulyaev [61]: path integrals are more sensitive to discretization issues than Langevin equations, and higher orders in Δt than those usually retained, eventually matter. This was also noted in [37, 62] in the quantum context, and further discussed in [20, 55, 63, 64].

4.3. A covariant action

If, instead of writing the infinitesimal action $\delta S_X^{\Delta t}$ using the Stratonovich convention as in equation (14), one uses the covariant discretization,

$$\delta S_X^{\Delta t} \stackrel{\beta_g}{=} \frac{\Delta t}{4D}\left[\frac{\frac{\Delta x}{\Delta t}-f(\bar{x}_0)}{g(\bar{x}_0)}\right]^2+\frac{\Delta t}{2}\left[f'(\bar{x}_0)-\frac{f(\bar{x}_0)g'(\bar{x}_0)}{g(\bar{x}_0)}\right], \tag{16}$$

where $\stackrel{\beta_g}{=}$ indicates that $\bar{x}_0$ is β_g-discretized as in equations (9) and (10). Compared to the standard Stratonovich scheme ($\beta_g\equiv 0$) one observes that equation (16) has less terms: the second line in equation (14) is now absent. This means that, in the $\Delta t\to 0$ limit, the covariant and the Stratonovich schemes are not equivalent when writing the action (while they are for the Langevin equation). This is a signature of the higher sensitivity of the path integral to the details of the discretization[6].

The two expressions we have obtained for the infinitesimal action, equations (14) and (16), are both valid, *and actually equal*, even though they lead to visually distinct continuous-time writings of the action (compare (15) with (19), below). Changing $\beta_g(x)$ in equation (9) modifies the continuous-time writing of the action (but not that of the Langevin equation). In the next section we prove that, in contrast to the Stratonovich case, the covariant discretization ensures the covariance of the action under a change of path $x(t)\mapsto u(t)=U(x(t))$ through the use of the chain rule.

We can draw here a helpful analogy: a multiplicative Langevin process can be described by equivalent but distinct continuous-time writings (depending on the discretization conventions). These are equally valid but only the Stratonovich one benefits from covariance. The same happens for the path-integral: the infinitesimal actions (14) and (16) (and their continuous-time writings (15) and (19)) are both correct but only (16) benefits from covariance.

4.4. The proof of covariance

For convenience we proceed backwards from u to x (see figure 2). The infinitesimal propagator for the process $u(t)$ reads

[6] Had we kept a term of the form $\gamma(x)\Delta x^3$ in the expansion of equation (9), this would not have changed the form (16) of $\delta S_X^{\Delta t}$ to the order relevant for the path integral (namely, up to $O(\Delta t)$ included). The covariant discretization (9) thus goes up to the optimal order in powers of Δx.

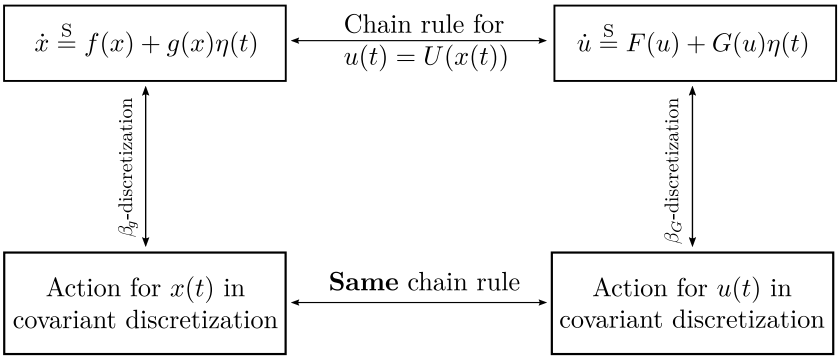

Figure 2. Schematic representation, for a change of variables $x \mapsto u(x)$, of how the covariant discretization scheme allows one to use the same rules of calculus for a Stratonovich-discretized Langevin equation and for their corresponding covariant Onsager–Machlup and MSRJD actions (19) and (21). Such a use of the chain rule would be incorrect in the traditional Stratonovich-discretized actions (15) and (22).

$$\mathbb{P}_U(u_1|u_0) \overset{\beta_G}{=} \frac{1}{\sqrt{4\pi D\Delta t}\,|G(u_1)|}\,\mathrm{e}^{-\frac{1}{2}\frac{\Delta t}{2D}\left[\frac{\frac{\Delta u}{\Delta t}-F(\bar{u}_0)}{G(\bar{u}_0)}\right]^2} \times \mathrm{e}^{-\frac{1}{2}\Delta t\left[F'(\bar{u}_0)-\frac{F(\bar{u}_0)G'(\bar{u}_0)}{G(\bar{u}_0)}\right]}. \tag{17}$$

We have to show that it yields back the infinitesimal propagator (13) and action (16) for the variable $x(t)$ after a generic change of variables.

First, using that $\mathbb{P}_X(x_1|x_0) = |U'(x_1)|\,\mathbb{P}_U(u_1|u_0)$, one notices that the prefactor of the propagator becomes the expected one, equation (13), for the variable $x(t)$, thanks to the end-point discretized prefactor. Then, the difficulty is to shift from the β_G-discretized variable $u(t)$ to the β_g-discretized variable $x(t)$, but this only requires a correct expansion at $O(\Delta t)$. With the recipe presented in the appendix A.2, one compares the following routes:

(A) in equation (17), express $\bar{u}_0$ as a function of $\bar{x}_0$ and Δx; expand in powers of $\Delta x = O(\Delta t^{1/2})$ up to order $O(\Delta t)$; and use the substitution rules (derived in [55] and recalled in appendix A.4) to handle powers of Δx of degree higher than 1;

(B) naively replace $\frac{\Delta u}{\Delta t}$ in equation (17) by $U'(\bar{x}_0)\frac{\Delta x}{\Delta t}$; $F(\bar{u}_0)$ by $U'(\bar{x}_0)f(\bar{x}_0)$; and $G(\bar{u}_0)$ by $U'(\bar{x}_0)g(\bar{x}_0)$.

Route (B) is in principle completely faulty because it misses many terms of orders $O(\Delta t^{1/2})$ and $O(\Delta t)$, as discussed in [55]. However, for the chosen β_g-discretization of equation (9), it correctly matches the outcome of route (A)—which happens to be the expected infinitesimal propagator $\mathbb{P}_X(x_1|x_0)$, given by equations (13) and (16). For other choices of time discretization, including the Stratonovich one, route (B) does not yield the correct result.

Since taking route (B) amounts to using the standard rules of calculus in the action, we have thus shown that, for the β_g-discretization (9), the correct rules of calculus in the infinitesimal propagator at small but finite Δt become identical to the standard rules of calculus in the continuous-time action when taking the $\Delta t \to 0$ limit. Showing the validity of the chain rule in this limit is simple for differentiable functions and significantly more intricate in a Langevin equation (where discretization issues matter), and it has demanded an even higher degree of caution inside the action, through the use of the covariant discretization (9) (see table 1).

Table 1. Minimal discretizations required for the chain rule of standard calculus to hold upon a change of variables $u(t) = U(x(t))$.

Situation	Required discretization
$x(t)$ is differentiable	Any can work
$x(t)$ is a Langevin process, equation (2)	Stratonovich, equation (5)
$x(t)$ is a path in the covariant action, equations (19) or (21)	Covariant, equation (9)
$x(t)$ is a path in the standard action, equations (20) or (22)	None works

4.5. Summary and continuous-time writing

Our main result is the direct construction of the probability of a time-discretized path. It takes the form of a path-integral probability (11) with an endpoint-discretized prefactor $\mathcal{N}[x]$ (equation (12)) and a β_g-discretized time-discrete action read from equation (16),

$$S_X^{\Delta t}[x] \overset{\beta_g}{=} \sum_{0\leqslant k<N} \left\{ \frac{1}{2}\frac{\Delta t}{2D}\left[\frac{\frac{\Delta x}{\Delta t} - f(\bar{x}_k)}{g(\bar{x}_k)}\right]^2 + \frac{\Delta t}{2}\left[f'(\bar{x}_k) - \frac{f(\bar{x}_k)g'(\bar{x}_k)}{g(\bar{x}_k)}\right]\right\}. \quad (18)$$

Taking the continuous-time limit, the path probability $\mathcal{N}[x]\mathrm{e}^{-S[x]}$ of a trajectory $[x(t)]_{0\leqslant t\leqslant t_\mathrm{f}}$ that evolves according to the Langevin equation (2) (understood in the Stratonovich sense) has an action given by

$$S_X[x] \overset{\beta_g}{=} \int_0^{t_\mathrm{f}} \mathrm{d}t \left\{ \frac{1}{4D}\left[\frac{\dot{x} - f(x)}{g(x)}\right]^2 + \frac{1}{2}f'(x) - \frac{1}{2}\frac{f(x)g'(x)}{g(x)} \right\} \quad (19)$$

which is a short-hand writing for the discrete expression (18). Such continuous-time writing of the action turns out to coincide with the result of [15, 16, 56]. In our formulation, it benefits from an essential feature: it is *covariant* under the change *of non-differentiable paths*, in the sense that the path probability of a process $u(t) = U(x(t))$ has a β_G-discretized action $S_U[u]$ that is inferred from the action equation (19) for $x(t)$ by merely passing from the variable x to u through the use of the standard chain rule of calculus, see figure 2. Such property is verified in the continuous-time writing of equation (19) (in a computation that is valid for differentiable paths); but its actual proof is done in discrete time, as presented in the previous paragraph, because the action we are interested in describes the path probability of non-differentiable trajectories, through equation (11).

Besides, if we were to read the action (19) as Stratonovich-discretized, the resulting expression would be incorrect: as directly checked, the summand of equation (18) evaluated for a Stratonovich-discretized $\bar{x}_k$ and a covariant-discretized $\bar{x}_k$ differ by non-constant terms of order Δt, that cannot be discarded in the $\Delta t \to 0$ limit. This explains why continuous-time derivations of the action (19), such as the one of Graham [16], are not amenable to an easy reconstruction of the path probability. For instance, in a subsequent work, Graham and Deininghaus [20] succeeded to do so, but at the price of multiplying the trajectory weight with a correction prefactor that is tuned in order to ensure covariance and probability conservation. In contrast, the construction we bring forward is self-contained and establishes that the covariant action (19) simply has to be read with the covariant discretization scheme.

5. Comparison to other path-integral constructions

5.1. Different approaches

To write the explicit path probability of a trajectory, the time-slicing procedure can be implemented in a variety of ways and, within the realm of stochastic processes, this was carried out in [7, 9, 11, 51, 58, 59]. The constructions proposed by these authors are not fully satisfactory as they all suffer from the same problem: their actions are neither covariant at the discretized nor at the continuous-time levels. A classical choice in these papers consists in writing the action in Stratonovich and discretizing the normalization prefactor $\mathcal{N}[x]$ in $\bar{x}_k$ (replacing $g(x_{k+1})$ by $g(\bar{x}_k)$ in (12)). This leads to the continuous-time writing

$$S[x(t)] \overset{\mathrm{S}}{=} \frac{1}{2}\int_0^{t_f} \mathrm{d}t \left\{ \frac{1}{2D}\left[\frac{\dot{x} - f(x) + D\,g(x)g'(x)}{g(x)}\right]^2 + f'(x) \right\} \tag{20}$$

which is not covariant under the standard rules of calculus. Several authors tried to cure this problem. We summarize these attempts, and why we think that the goal was not fully achieved, in the next paragraph. Fox also put forward a path-integral construction that relies on considering a colored instead of a white noise, with a finite correlation time τ [65, 66]. Although this approach has the advantage to handle more regular paths, the $\tau \to 0$ limit yields back a Stratonovich-discretized action and is not covariant.

5.2. Covariant approaches

The first important progress in solving this problem is due to Stratonovich [56, 57], who constructed a covariant continuous-time action, whose writing is the same as equation (19). Horsthemke and Bach [15] and Graham [16] independently derived the same action in one dimension, and Graham further achieved the same program in dimension larger than one. What they did was to build a path integral with an action expressed in continuous time that is consistent with the underlying Langevin equations, and that can be blindly manipulated with the usual rules of differential calculus, as if the paths were differentiable. However their construction of the path integral is built from locally optimal differentiable paths. The action (19) thus bears different meanings in the mentioned references and in this work. Related works in mathematics have made such an approach rigorous. Either using changes of path probability [67, 68] or more direct techniques (see the work of Takahashi and collaborators [21, 22] and of Capitaine [23]), the idea is to determine the most probable path[7] going from one point to an other as extremizing an Onsager–Machlup covariant action. Such constructions are possible but do not provide the path probability of an arbitrary non-differentiable trajectories (which is the aim of our theoretical physicist's construction).

In the immediate aftermath of Graham's result, the search for an ambiguity-free definition of the path probability of a trajectory began. This is commented by Graham in [17], and it triggered more works [6, 18–20, 55, 63, 64, 69–72] in the direction of finding a proper discretization, the continuum limit of which would fall back on the action (19). This problem was not solved until this paper: we have found for this action an explicitly discretized picture that plays an analogous role to the Stratonovich rule (equation (5)) for Langevin equations. In the context of quantum mechanics in curved spaces, DeWitt [35] followed a construction where the action is evaluated along a succession of infinitesimal optimal trajectories that obey Euler–Lagrange equation—a construction that also has been made rigorous by mathematicians

[7] A 'path' seen as an infinitesimal tube around a *differentiable* trajectory.

[73, 74]. Such a procedure yields the same visual limit as the action (19) but endows it with a completely different meaning.

The covariant discretization that we propose in fact provides a step towards extending stochastic calculus to path integrals, by defining the time integral of the action (19) through a procedure generalizing the usual stochastic integral. From our physicist's viewpoint, stochastic calculus provides a definition of the integral $\int \mathrm{d}t\,[A(x)+B(x)\dot{x}]$ in a limiting procedure that involves a careful choice of discretization, together with being compatible with continuous-time rules of differential calculus (the standard chain rule). The construction we put forward allows one to do the same for $\int \mathrm{d}t\,[A(x)+B(x)\dot{x}+C(x)\dot{x}^2]$ inside an exponential.

6. Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) path-integral formulation

Since the early formulation of quantum mechanics in terms of path integrals, there have been two equivalent expressions for the transition amplitudes. One, that we have just discussed extensively, involves a single position field. An alternative one involves an additional conjugate momentum field. The latter can be removed or included at will by Gaussian integration. A mirror image of the auxiliary momentum field exists for stochastic dynamics: the alternative to the original Onsager–Machlup formulation is the MSRJD approach [9, 11–13, 75] and involves an additional so-called response field. The purpose of this section is to extend our findings to this formalism. Again, we adopt the language of stochastic dynamics, but our results equally apply to quantum mechanics.

6.1. Continuous-time MSRJD covariant action

In the MSRJD approach one introduces a response field $\hat{x}(t)$ to represent the trajectory weight in a manner that allows one, for instance, to get rid of some non-linearities of the action (19). Physics-wise, this setting facilitates the study of correlations and response functions on an equal footing, and to linearize (to some extent) possible symmetries of the process under scrutiny (time-reversal, rapidity reversal, etc). We now present our result for the covariant MSRJD action before describing its construction and its full time-discrete implementation.

In the covariant discretization scheme of equation (9), the action

$$S[\hat{x},x] \overset{\beta_g}{=} \int_0^{t_f} \mathrm{d}t \left\{ \hat{x}\big(\dot{x}-f(x)+D\,g(x)g'(x)\big) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) - \frac{-D}{4}g'(x)^2 - \frac{1}{2}\frac{g'(x)}{g(x)}\,\dot{x} \right\} \tag{21}$$

describes the path probability measure as $\mathcal{D}x\,\mathcal{D}\hat{x}\,\mathrm{e}^{-S[\hat{x},x]}$. In this path integral one can directly change variables covariantly using the standard chain rule and avoiding any Jacobian contribution. In continuous time, this property is tediously checked by direct computation using the chain rule of calculus together with the correspondence $\hat{x}(t) = U'(x(t))\,\hat{u}(t)$ between response fields. In contrast, the historically derived MSRJD action in Stratonovich discretization reads

$$\int_0^{t_f} \mathrm{d}t \left\{ \hat{x}\big(\dot{x}-f(x)+D\,g(x)g'(x)\big) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) \right\} \tag{22}$$

and applying the chain rule to it leads to inconsistencies [60].

6.2. Discretized MSRJD action

To construct the MSRJD representation, one rewrites the infinitesimal propagator for $x(t)$ (equations (13) and (16)) by using at every time step a Hubbard–Stratonovich transformation of the form $\sqrt{2\pi/a}\,\mathrm{e}^{-\frac{1}{2}\frac{b^2}{a}} = \int_{i\mathbb{R}} \mathrm{d}\hat{x}\,\mathrm{e}^{\frac{1}{2}a\hat{x}^2 - b\hat{x}}$ for the following choice of parameters a and b

$$a = 2Dg(\bar{x}_t)^2\,\Delta t, \quad b = \left[\frac{\Delta x}{\Delta t} - f(\bar{x}_t)\right]\Delta t, \tag{23}$$

which gives

$$\mathbb{P}(x_1|x_0) \stackrel{\beta_g}{=} \left|\frac{g(\bar{x}_0)}{g(x_1)}\right| \int_{i\mathbb{R}} \mathrm{d}\hat{x}_0\,\mathrm{e}^{-\delta S[\hat{x}_0,\bar{x}_0]}, \tag{24}$$

$$\delta S[\hat{x}_0,\bar{x}_0] \stackrel{\beta_g}{=} \Delta t\left\{\hat{x}_0\left[\frac{\Delta x}{\Delta t} - f(\bar{x}_0)\right] - Dg(\bar{x}_0)^2\hat{x}_0^2 + \frac{1}{2}f'(\bar{x}_0) - \frac{1}{2}\frac{g'(\bar{x}_0)}{g(\bar{x}_0)}f(\bar{x}_0)\right\}, \tag{25}$$

which completely encodes the continuous-time expression[8]

$$\tilde{S}[\hat{x},x] \stackrel{\beta_g}{=} \int_0^{t_f} \mathrm{d}t\left\{\hat{x}\left(\dot{x} - f(x)\right) - Dg(x)^2\hat{x}^2 + \frac{1}{2}f'(x) - \frac{1}{2}\frac{g'(x)}{g(x)}f(x)\right\}. \tag{26}$$

Up to a translation of the field $\hat{x}(t)$ by $g'/(2g)$, one recovers equation (21). The symbol β_g over the equality sign means that functions of the variable x are β_g-discretized, i.e. evaluated at $\bar{x}_k$. The field $\hat{x}(t)$ is not discretized in the same way as the field $x(t)$ is: a variable $\hat{x}_t$ is introduced at each t and merely associated to $\bar{x}_k$. The proof of the covariance presents more intricate issues than for the Onsager–Machlup action, and is sketched in appendix A.3.

7. Summary and outlook

When dealing with fluctuating signals as encountered in quantum mechanics or stochastic processes, whose shared trait is non-differentiability, physicists rely on a triptych of methods: solving a linear problem involving an operator (Schrödinger or Fokker–Planck equations), resorting to stochastic calculus (Langevin equations), or using path integrals (field theory). As we have discussed, there is a vast number of operations for which path integrals have been known to be badly flawed. This surely explains why path integrals never became a tool of choice for mathematicians working on similar problems. What we have shown in the present work is how to construct a path-integral calculus that directly manipulates physical paths and that is devoid of what we view as its biggest flaw. It is our belief that our proposed construction should not only trigger a revival of interest on the mathematics side, but also on the physics one. Mathematics-wise, though we would not blush with embarrassment about our physicist's

[8] Note from equation (24) the appearance, in the discretized expression for the probability of a path, of a normalization prefactor $\mathcal{N}_{\text{MSR}}[x(t)] = \prod_{0\leqslant k<N}\left|\frac{g(\bar{x}_k)}{g(x_{k+1})}\right|$ in front of the exponential weight. This $\mathcal{N}_{\text{MSR}}$ warrants that a change of path in the action (21) induces no spurious contribution coming from the Jacobian $|U'(x_1)|$ in $\mathbb{P}_X(x_1|x_0) = |U'(x_1)|\,\mathbb{P}_U(u_1|u_0)$.

derivation, it is almost certain that many more steps are needed to bring our building of covariant path integrals on a rigorous par with other aspects of stochastic calculus. Physics-wise, we see immediate consequences, and open questions. Among the former, given the pedagogical importance of path integrals in higher education, we would advocate strongly in favor of our presentation (which time and efforts will surely smoothen and hopefully simplify) rather than in existing ones which suffer from well-known problems. Second, given the lack of control, so far, in nonlinear manipulations of fields, which have been put to work in so many areas, it seems like a necessity to return to these and sort out whether and how path-integral based results are altered by taking our corrected formalism into account. Transformations of the action based on the chain rule, as simple as integrations by parts for instance, are in principle forbidden unless one uses the covariant discretization. This is especially important in areas of physics where no alternative to path integrals exist (like in path-integral based quantization issues). This brings us to future research directions, which we briefly list: What about higher space dimensions?, What about supersymmetries?, What about field theories expressed in second quantized form with coherent-states fields?

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Appendix

A.1. An exact covariant discretization of the Langevin equation

Since the path-integral formulation requires higher orders in Δt than usually, it appears crucial to find a discretization scheme that is consistent with the chain rule to a high-enough order. Fortunately, such a scheme can be found, and this is one of the main results in this paper. The inspiration comes from the field of calculus with Poisson point processes [76–79], though our solution departs from anything that has already been proposed. We postulate that equation (2) is to be understood in the form

$$\Delta x = \mathbb{T}_{f,g} f(x(t))\Delta t + \mathbb{T}_{f,g} g(x(t))\Delta\eta \tag{A.1}$$

with $\Delta\eta = \eta\Delta t$ and where the operator $\mathbb{T}_{f,g}$ acts on an arbitrary function h as

$$\mathbb{T}_{f,g} h(x) = \frac{\mathrm{e}^{\mathcal{D}(x)\,\frac{\mathrm{d}}{\mathrm{d}x}} - \mathbf{1}}{\mathcal{D}(x)\,\frac{\mathrm{d}}{\mathrm{d}x}} h(x) = \sum_{n\geqslant 0} \frac{\left(\mathcal{D}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)^n}{(n+1)!} h(x)\,. \tag{A.2}$$

Here[9] $\mathcal{D}(x) = f(x)\Delta t + g(x)\Delta\eta$ acts as an operator, and it does not commute with $\frac{\mathrm{d}}{\mathrm{d}x}$. When acting on f the operator $\mathbb{T}_{f,g}$ leaves us with a complicated function of both $x(t)$ and $\Delta\eta$, which, in an implicit fashion through equation (A.1), is then a function of $x(t)$ and $\Delta x = x(t+\Delta t) - x(t)$. As is perhaps less obvious than in previous discretization schemes, the $\Delta t \to 0$ limit also gets us back to equation (2). This is because $\Delta\eta$, which is of order $\Delta t^{1/2}$, also goes to 0. We remark here that truncating the sum at $n = 2$ in (A.2) one recovers an expression that is close to the Milstein

[9] In the study of Poisson point-processes with multiplicative noise, the appropriate discretization restricts to $\mathcal{D}(x) = g(x)\Delta\eta$, but in our context the supplemental term $f(x)\Delta t$ is needed.

[44, 45] scheme used in numerical simulations of Langevin equation (one has to discard a term $\propto \Delta t\, f(x)\, f'(x)$ and switch from Stratonovich to Itō calculus).

The complex appearance of this discretization rule (A.1) and (A.2) should not conceal its central property: it is consistent with the chain rule *for any finite* Δt. In other words, when the evolution of x is understood with equation (A.1), one can manipulate a function $u(t) = U(x(t))$ as if it were differentiable, and $\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t} = \dot{x}\, U'(x)$ holds in the sense that

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} = \mathbb{T}_{F,G}F(u(t)) + \mathbb{T}_{F,G}G(u(t))\Delta\eta, \tag{A.3}$$

where $F(u)$ and $G(u)$ are the force and the noise amplitude of the Langevin equation verified by $u(t)$, defined as $F(U(x)) = U'(x)f(x)$ and $G(U(x)) = U'(x)g(x)$.

The unpleasant feature of the discretization rule in equation (A.2) is that it is expressed in terms of $\Delta\eta$ rather than in terms of Δx, as we did in equation (5). This means that equation (A.2) cannot be used as such in the definition of the path integral in which the noise $\eta(t)$ is eliminated in favor of $x(t)$. We would rather express equation (A.1) in terms of a function $\delta(\Delta x)$ such that

$$\mathbb{T}_{f,g}h(x) = h(x + \delta(\Delta x)). \tag{A.4}$$

An expansion of δ in powers of Δx can be found:

$$\delta(\Delta x) = \alpha\Delta x + \beta(x)\Delta x^2 + \dots \tag{A.5}$$

where $\alpha = \frac{1}{2}$, $\beta = \beta_g = \frac{1}{24}\frac{g''}{g'} - \frac{1}{12}\frac{g'}{g}$, etc. We shall henceforth keep the functional dependence of these functions on g explicit. Keeping in mind that $\Delta x = O(\Delta t^{1/2})$ as $\Delta t \to 0$, at minimal order $\delta(\Delta x) = \frac{1}{2}\Delta x$ and we recover the Stratonovich discretization (5), for which the chain rule in equation (A.3) is valid with up to an error of order $\Delta t^{1/2}$. Including the β term in equation (A.5) with $\beta = \beta_g$ renders the error of order Δt (and so on and so forth when increasing the order of the expansion). Terms of order higher than β in (A.5) will prove useless for our purpose. This is the discretization scheme that we adopted in equations (9) and (10) in the time-slicing procedure involved in constructing our formulation of the path integral.

A.2. Changing variables while respecting the discretization

We explain here the methodology used to manipulate the infinitesimal propagator in the small Δt limit, following [55]. When passing from one infinitesimal propagator to another, one needs to reconstitute the β_g-discretization of the variable $x(t)$ in $\mathbb{P}_X(x_1|x_0)$ (equations (13) and (16)) from the β_G-discretization of the variable $u(t)$ in $\mathbb{P}_U(u_1|u_0)$ (equation (17)). The idea is to express the time-discrete values $u_0 = U(x_0)$, $u_1 = U(x_1)$ and $\bar{u}_0$ appearing in the r.h.s. of equation (17) as a function of $\bar{x}_0$ and Δx, using

$$\begin{aligned} \bar{u}_0 &= U(x_0) + \frac{1}{2}[U(x_1) - U(x_0)] + \beta_G(U(x_0))\,[U(x_1) - U(x_0)]^2, \\ x_0 &= \bar{x}_0 - \frac{1}{2}\Delta x - \beta_g(\bar{x}_0)\,\Delta x^2, \end{aligned} \tag{A.6}$$

$$x_1 = \bar{x}_0 + \frac{1}{2}\Delta x - \beta_g(\bar{x}_0)\,\Delta x^2. \tag{A.7}$$

The strategy is the following: first, use these expressions in equation (17); second, expand this equation in powers of Δt and Δx, keeping in mind that the latter is $O(\Delta t^{1/2})$. The result takes the form

$$\frac{\mathcal{N}}{|g(x_1)|}\,\mathrm{e}^{-\frac{1}{2}\frac{\Delta x^2}{2Dg(\bar{x}_0)^2\Delta t}} \times \big[1 + \text{polynomial in } \Delta x \text{ and } \Delta t\,\big]. \tag{A.8}$$

The fraction in the exponential is $O(\Delta t^0)$ and cannot be expanded; in fact, it defines the dominant order $O(\Delta t^{1/2})$ of Δx. The polynomial contains terms of the form $\Delta t^n \Delta x^m$ which are of order $O(\Delta t^{1/2})$ and $O(\Delta t)$. Higher-order terms ($O(\Delta t^{3/2})$ and higher) can be discarded because they do not contribute to the action. Many of the terms in the polynomial do not present an obvious $\Delta t \to 0$ limit (e.g. $\Delta x^4\,\Delta t^{-1}$) but the substitution rules derived in [55] allow one to take the continuous-time limit. For completeness, these are recalled (and slightly reformulated) in appendix A.4. The last stage of the procedure consists in reexponentiating the resulting factor $[1 + \ldots]$ obtained from equation (A.8). One then recovers the expected propagator $\mathbb{P}_X(x_{\Delta t}|x_0)$ of equations (13) and (16) as announced.

The same procedure allows one to change variables in the historical action (20) but this involves rules of calculus sharing little kindred with the standard ones (see [55]). The covariant discretization, instead, yields back the usual chain rule as $\Delta t \to 0$.

A.3. Sketch of the derivation of the covariance of the MSRJD path-integral

The actual derivation of the covariance property involves a careful handling of the time-discrete infinitesimal propagator, by analyzing the contributions that arise order by order in powers of Δt upon the change of variables $U(t) = u(x(t))$.

One proves that only for the covariant discretization it is valid to naively change variables in equations (24) and (25): namely, going from the fields $(\hat{u}, u)$ to $(\hat{x}, x)$, one can replace $\frac{\Delta u}{\Delta t}$ by $U'(\bar{x}_0)\frac{\Delta x}{\Delta t}$, $F(\bar{u}_0)$ by $U'(\bar{x}_0)f(\bar{x}_0)$, and $G(\bar{u}_0)$ by $U'(\bar{x}_0)g(\bar{x}_0)$. Such operations, combined with $\hat{u}_0 = \hat{x}_0/U'(\bar{x}_0)$, would normally yield an incorrect result by missing essential contributions of order $O(\Delta t^{1/2})$ and $O(\Delta t)$. Satisfactorily, these manipulations are correct for our chosen covariant discretization. The proof follows a procedure similar to the one we presented for the Onsager–Machlup case by comparing a correct route (A) with a naive route (B), with three important caveats: (i) One has to pay attention to the fact that $\hat{x}_t \sim \Delta t^{-1/2}$ at every time step, as inferred from the scaling of a in the Hubbard–Stratonovich transform (23), implying that the expansions in powers of $O(\Delta t)$ bring in terms that one can be tempted to throw away at first sight; (ii) One has to design additional substitution rules in order to handle powers of $\hat{x}_0$ larger than 1. This is done following a procedure similar to the one of [55] (see appendix A.4); (iii) Unexpectedly, in contrast to the Onsager–Machlup case exposed previously, the prefactor $\left|\frac{g(\bar{x}_0)}{g(x_1)}\right|$ in (24) *brings a Jacobian contribution into the action* upon the time-discrete change of variables of route (A), which compensates precisely a term that is missing when naively substituting $\frac{\Delta u}{\Delta t}$ by $U'(\bar{x}_0)\frac{\Delta x}{\Delta t}$ along route (B).

To summarize, we have shown that changing variables in the MSRJD action (21) can be done following the standard rules of differential calculus, provided that the discrete-time construction of the path-integral weight is performed according to the covariant discretization of equations (9) and (10)—leading to a modified action as compared to the historical Stratonovich-discretized one.

A.4. Substitution rules

Denoting $\lceil \Delta x^2 \rfloor = 2Dg(\bar{x}_0)^2\Delta t$, the substitution rules deduced in [55] can be reformulated as follows

$$\Delta x^2 \mapsto \lceil \Delta x^2 \rfloor, \tag{A.9}$$

$$\Delta x^3\, \Delta t^{-1} \mapsto 3\, \Delta x\, \lceil \Delta x^2 \rfloor\, \Delta t^{-1}, \tag{A.10}$$

$$\Delta x^4\, \Delta t^{-1} \mapsto 3\, \lceil \Delta x^2 \rfloor^2\, \Delta t^{-1}, \tag{A.11}$$

$$\Delta x^6\, \Delta t^{-2} \mapsto 15\, \lceil \Delta x^2 \rfloor^3\, \Delta t^{-2}. \tag{A.12}$$

Note that, as discussed in [55], the substitution rule (A.10) *cannot* be used inside the exponential of the infinitesimal propagator; indeed, since $\Delta x^3\, \Delta t^{-1} = O(\Delta t^{1/2})$ one has $e^{h(x)\Delta x^3\, \Delta t^{-1}} = 1 + h(x)\Delta x^3\, \Delta t^{-1} + \frac{1}{2}[h(x)\Delta x^3\, \Delta t^{-1}]^2 + O(\Delta t^{3/2})$ and the second term of this expansion would be wrong if one had first applied the rule (A.10) and then expanded. This is the trivial but shrouded reason why the procedure exposed in appendix A.2 has to be performed by expanding the terms of order $\Delta t^{>0}$ outside of the exponential of the infinitesimal propagator of (A.8). This reflects the fact, known to mathematicians, that the validity of the continuous-time chain rule is relatively weak, even in the Stratonovich discretization: it cannot be manipulated without care by, for instance, taking its square and exponentiating it—as one would do by naively using it in the Onsager–Machlup action. For further discussion on this subject, see [55].

ORCID iDs

Leticia F Cugliandolo https://orcid.org/0000-0002-4986-8164
Vivien Lecomte https://orcid.org/0000-0003-4025-5852
Frédéric van Wijland https://orcid.org/0000-0002-3335-8573

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