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Effective driven dynamics for one-dimensional conditioned Langevin processes in the weak-noise limit

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Abstract. In this work we focus on fluctuations of time-integrated observables for a particle diffusing in a one-dimensional periodic potential in the weak-noise asymptotics. Our interest goes to rare trajectories presenting an atypical value of the observable, that we study through a biased dynamics in a large-deviation framework. We determine explicitly the effective probability-conserving dynamics which makes rare trajectories of the original dynamics become typical trajectories of the effective one. Our approach makes use of a weak-noise path-integral description in which the action is minimised by the rare trajectories of interest. For 'current-type' additive observables, we find criteria for the emergence of a propagative trajectory minimising the action for large enough deviations, revealing the existence of a dynamical phase transition at a fluctuating level, whose singular behaviour is between first and second order. In addition, we provide a new method to determine the scaled cumulant generating function of the observable without having to optimise the action. It allows one to show that the weak-noise and the large-time limits commute in this problem. Finally, we show how the biased dynamics can be mapped in practice to an explicit effective driven dynamics, which takes the form of a driven Langevin dynamics in an effective potential. The non-trivial shape of this effective potential is key to understand the link between the dynamical

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phase transition in the large deviations of current and the standard depinning transition of a particle in a tilted potential.

Keywords: fluctuation phenomena, large deviations in non-equilibrium systems, stochastic particle dynamics

Contents

1. Introduction

Traditional approaches in statistical physics are based on the study of the probability distribution of microscopic configurations at a given time [1]. Although such approaches have been very successful at equilibrium where configurations with the same energy are distributed uniformly in an isolated system, one is faced with difficulties when considering the statistics of configurations in non-equilibrium steady-states, as this statistics is in general non-uniform and unknown. It has been realised in the last decades that a more general space-time formulation, which deals with the statistics of full trajectories (that is, configurations as a function of time on a large time window) could be formulated in a quite general way, even for non-equilibrium systems [2, 3]. Moreover, the large-deviation formalism provides an efficient framework to formulate the problem [4–8]. The large-deviation formalism is particularly useful for instance to evaluate the statistics of time-integrated observables (e.g. particle current or dynamical activity), which are natural observables when characterising the statistics of trajectories [7–16]. One can for instance consider a modified equilibrium statistics of trajectories conditioned to a given value of a time-integrated observable, like the average particle current. It is then of interest to ask whether this artificial, biased dynamics shares some similarities with (or even could be mapped to) a 'real' non-equilibrium dynamics. In other words, does a physical force which drives a system into a non-equilibrium state (and thus generates a given current) select all trajectories having a given average current in a least-biased way?

In practice, fixing a given average value of an integrated observable is done by introducing a conjugated Lagrange multiplier, in the same way as, at equilibrium, temperature fixes the average energy in the canonical ensemble [7]. This Lagrange multiplier enters the definition of a 'deformed' Markov operator that describes the biased dynamics. A well-know difficulty is that this deformed Markov operator no longer conserves probability, and cannot straightforwardly be interpreted as describing a *bona fide* probability-preserving dynamics. It has however been shown [17–20] how a relatively simple but abstract transformation of the deformed Markov operator allows one to define a closely related probability-conserving Markov operator, which defines an 'effective dynamics' that is *asymptotically equivalent* at large times to the biased dynamics and the conditional dynamics [20–22] after proper normalisation.

With these ideas in mind, the results of this paper are the following: focusing on the example of a particle diffusing in a periodic potential in one-dimension, we make analytical progress in the determination of large-deviation functions (LDF) quantifying the distribution of generic additive observables. We show that the two asymptotic regimes we are considering, namely, the large-time and small-noise ones, can be taken in any order. A standard variational principle arising from a weak-noise, Wentzel–Kramers–Brillouin (WKB) [23] type asymptotic analysis is partially solved analytically and replaced by a much simpler one. This allows us to obtain an explicit form of the effective dynamics, and to study the occurrence of a singularity of the LDF, which corresponds to a 'dynamical phase transition' separating different regimes of fluctuations.

LDFs of the distribution of additive observables in such periodic one-dimensional diffusive problems have been the subject of a number of studies in the past years, but the actual derivation of LDFs were mostly limited to peculiar additive observables, such as the entropy production [24–26] or the current [27–29], and were not fully explicit analytically. In this paper, we extend a recent work in which the large deviations of the current were studied in the weak-noise asymptotics [29] to the case of generic time-integrated observables. Instead of relying on a numerical analysis of a truncated Fourier–Bloch decomposition of a spectral problem underlying the LDF problem (as done in [25, 29]), our work is based on an analytical study of a variational principle: we find explicit generic solutions to the variational problem that governs the value of LDFs, using a different approach than the one presented in [27, 28] (a detailed comparison is provided when presenting our results). Our approach also leads to results that present a broader interest: the variational principle that determines the

LDF implies to find 'optimal trajectories' which minimise an action, as in Lagrangian mechanics; such trajectories are indexed by a conserved quantity (the energy) which, in our LDF problem, also has to be optimised over—in contrast to other physical situations where the energy is given. We show that the value of the action for the trajectory with optimal energy takes a very special form that simplifies the actual computation of the LDF, and that could prove useful in other contexts where variational principles fit in the framework of Lagrangian mechanics.

When considering the large-deviation scaling of the steady-state distribution in the weak-noise limit, the appearance of singularities (as non-differentiabilities) is known to occur in the quasi-potential of non-equilibrium dynamics [30–32]. For the distribution of time-integrated observables, the occurrence of *another* type of singularities has been reported during the last decade in varied systems [10, 12, 33–38]. These later singularities correspond to the type of dynamical phase transitions we are interested in in this paper. They describe how the trajectories that lead to an atypical value of the time-integrated observable can change from one class to another when varying the value of this observable, in the small-noise asymptotics. If the occurrence of dynamical phase transitions in periodic 1D diffusion problems has been analysed in previous studies [26, 27, 29], we provide in this paper a thorough analysis of the competition between time-dependent and time-independent typical trajectories on both sides of the transition, together with a complete analysis of the LDF singularity (which takes the form of a first-order transition with a logarithmic prefactor that makes it continuous instead of discontinuous). The occurrence of transitions of such form had not been determined previously in this context, to our knowledge.

The paper is organised as follows. In section 2, we define the Langevin dynamics and reformulate it in a path-integral framework, to be able to bias the dynamics by a given value of the integrated additive observable considered. We also introduce the large deviation form of the action at large time, leading to the definition of the scaled cumulant generating function. These first two sections are largely expository and allow us to describe the problem we study in a self-contained manner. In section 3, we show how the scaled cumulant generating function can be evaluated explicitly, in the small noise limit, using a saddle-point calculation. Finally, in section 4, we use the knowledge of the scaled cumulant generating function to derive an effective physically driven dynamics that leads to the same statistics of trajectories (after normalisation), and discuss its interpretation.

2. Rare trajectories: conditioning or biasing the dynamics

We present in this section the Langevin dynamics we focus on, and the type of the observables whose distribution we are interested in. We refer the reader to existing reviews [8, 39] for generalisations for instance to mixed Langevin and Markov jump processes.

2.1. Langevin dynamics and additive observables

Consider a particle of position $x(t)$ at time t subjected to a force $F(x)$ and a thermal noise $\eta(t)$. In the overdamped limit, the evolution of its position is described by the Langevin equation

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$$\dot{x}(t) = F(x(t)) + \sqrt{\varepsilon}\,\eta(t)\,, \tag{1}$$

where $\eta(t)$ is a Gaussian white noise of average $\langle \eta(t) \rangle = 0$ and correlation function $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$. Our interest goes to observables depending on the trajectory on a time window $[0, t_\mathrm{f}]$ and taking the form[4]

$$A(t_\mathrm{f}) = \int_0^{t_\mathrm{f}} h(x(t))\,\mathrm{d}t + \int_0^{t_\mathrm{f}} g(x(t))\,\dot{x}(t)\,\mathrm{d}t\,. \tag{2}$$

Examples of such an *additive* observable encompass time-integrated current, work, entropy production, or activity for specific choices of the functions $h(x)$ and $g(x)$ (see [39] for examples). Contributions involving the function h are termed to be of 'density-type' while the ones involving g are of the 'current-type'. One is interested in the distribution $P(A, t_\mathrm{f})$ of the observable A at time t_f, in the weak-noise and/or the large-time limit. In these asymptotics, the scaling form of $P(A, t_\mathrm{f})$ is described within the framework of the large-deviation theory, which has witnessed a tremendous development in the last decades both in mathematics (within the Donsker–Varadhan approach [41–44], the Gärtner–Ellis theorem [45, 46] and the Freidlin–Wentzell framework [47]) and in statistical physics in the study of many example systems (see for instance [10, 13, 35, 48–50]). While we refer the reader to many existing reviews for a general presentation of this approach [4–8, 39], we provide here for completeness a self-contained presentation of the tools used in our context.

The Langevin equation (1) is equivalently described by the Onsager–Machlup weight of a trajectory $[x(t)]_{0 \leqslant t \leqslant t_\mathrm{f}}$ of duration t_f

$$\mathcal{P}[x(t), t_\mathrm{f}] \propto \mathrm{e}^{-\frac{1}{2\varepsilon}\int_0^{t_\mathrm{f}} (\dot{x}-F)^2\,\mathrm{d}t} \tag{3}$$

(valid in the weak-noise asymptotics $\varepsilon \to 0$, since we are working with the Stratonovich convention) or by the Fokker–Planck equation for the evolution of the probability $P(x, t)$ of finding the particle at a position x at time t, which takes the form

$$\partial_t P(x, t) = \mathbb{W} P(x, t). \tag{4}$$

The Fokker–Planck operator $\mathbb{W}$ reads:

$$\mathbb{W}\cdot = -\partial_x(F(x)\cdot) + \frac{1}{2}\varepsilon\partial_x^2 \cdot\ . \tag{5}$$

Note that the conservation of probability reads $\langle -|\mathbb{W} = 0$ where $\langle -|$ is the flat vector with all components equal to 1 (i.e. $\langle -|x\rangle = 1$ for all x)[5]. In other words, $\langle -|$ is a left eigenvector of $\mathbb{W}$ of eigenvalue 0. We now recall how, by studying the generating function of the observable A, one can extend the operator approach we just presented in order to study the distribution of A.

[4] The stochastic integral in (2) is taken in the Stratonovich convention (see e.g. [40] for a review on stochastic integrals).

[5] Here we use a bra-ket notation to describe the vector space on which operators such as $\mathbb{W}$ act, with $|x\rangle$ the state representing the particle at position x and $\langle x|$ its transpose. These define the canonical scalar product $\langle x|x'\rangle = \delta(x - x')$.

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2.2. Path-integral and Fokker–Planck representations

We aim at characterising the physical features of trajectories $[x(t)]_{0\leqslant t\leqslant t_{\rm f}}$ presenting an arbitrary (for instance, atypical) value of the observable, A. One way to proceed is to determine the probability $P(x,A,t_{\rm f})$ of the particle to be in position x at time t_f, while having observed a value A of the additive observable (2) on the time window $[0,t_{\rm f}]$. It reads as follows

$$P(x,A,t_{\rm f}) = \left\langle \int_{x(0)}^{x(t_{\rm f})=x} \mathcal{D}x\, \delta(A-A(t_{\rm f}))\, {\rm e}^{-\frac{1}{2\varepsilon}\int_0^{t_{\rm f}}(\dot{x}-F)^2\,{\rm d}t} \right\rangle_{x(0)} \tag{6}$$

where the notation $\langle\dots\rangle_{x(0)}$ indicates an average over the initial position $x(0)$ with a distribution $P_{\rm i}(x)$. We implicitly assume that $x(0)$ is distributed with $P_{\rm i}(x)$ in the following, except otherwise indicated.

It is difficult in general to determine $P(x,A,t_{\rm f})$ or even to write a closed equation for this 'microcanonical' probability. Following Varadhan [51], one performs a Laplace transform and introduces the following generating function (and its associated *biased* ensemble)

$$\hat{P}(x,\lambda,t_{\rm f}) = \int {\rm e}^{-\frac{\lambda}{\varepsilon}A}\, P(x,A,t_{\rm f})\,{\rm d}A \tag{7}$$

$$= \left\langle \int_{x(0)}^{x(t_{\rm f})=x} \mathcal{D}x\, {\rm e}^{-\frac{1}{\varepsilon}\left[\lambda A(t_{\rm f})+\frac{1}{2}\int_0^{t_{\rm f}}(\dot{x}-F)^2\,{\rm d}t\right]} \right\rangle_{x(0)} \equiv \left\langle \int_{x(0)}^{x(t_{\rm f})=x} \mathcal{D}x\, {\rm e}^{-\frac{1}{\varepsilon}S_\lambda[x(t),t_{\rm f}]} \right\rangle_{x(0)} \tag{8}$$

where $S_\lambda[x(t),t_{\rm f}]$ is defined as

$$S_\lambda[x(t),t_{\rm f}] = \lambda A(t_{\rm f}) + \frac{1}{2}\int_0^{t_{\rm f}}(\dot{x}-F)^2\,{\rm d}t = \int_0^{t_{\rm f}} \left\{ \frac{1}{2}(\dot{x}-F)^2 + \lambda\big(h+\dot{x}g\big) \right\} {\rm d}t \tag{9}$$

with $F\equiv F\big(x(t)\big)$, $g\equiv g\big(x(t)\big)$ and $h\equiv h\big(x(t)\big)$ to lighten notations. In analogy with thermodynamics, this defines a 'canonical' version of the problem, where trajectories are biased by an exponential factor ${\rm e}^{-\frac{\lambda}{\varepsilon}A(t_{\rm f})}$ on the time window $[0,t_{\rm f}]$. In the large $t_{\rm f}$ limit, as detailed below, the joint distribution corresponding to (6) and the generating function (7) present the same LDF scaling provided that the value of λ is well chosen as a function of A, as in any change of ensemble[6] (see e.g. [8] for a review). It is known that the evolution in time of $\hat{P}(x,\lambda,t_{\rm f})$ reads $\partial_t\hat{P} = \mathbb{W}_\lambda\hat{P}$ with a biased Fokker–Planck operator given by a generalised Feynman–Kac formula [21, 22]

$$\mathbb{W}_\lambda\cdot = -\partial_x\big((F-\lambda g)\,\cdot\,\big) + \frac{1}{2}\varepsilon\partial_x^2\cdot + \frac{\lambda}{\varepsilon}\left(\frac{\lambda}{2}g^2 - gF - h\right)\cdot\,. \tag{10}$$

We provide here for completeness an alternative derivation based on path integrals, as it also sheds light on how one can jointly change process and LDF observable while keeping the same action. The starting point consists in remarking that the biased action $S_\lambda[x(t),t_{\rm f}]$ in (9) is equivalently written as:

[6] This implies some requirement on the convexity of a large deviation function, as we explain below.

$$S_\lambda[x(t),t_{\rm f}] = \int_0^{t_{\rm f}} \frac{1}{2}(\dot{x} - F + \lambda g)^2\, {\rm d}t \quad - \int_0^{t_{\rm f}} \lambda\Big(\frac{\lambda}{2}g^2 - gF - h\Big)\, {\rm d}t. \tag{11}$$

This rewriting (11) of the action (9) is only the factorisation of the $\lambda\dot{x}g$ contribution into the square term of the action. The action given in equation (11) can be interpreted as follows. The first integral in (11) is the action of a modified process[7] $x(t)$ obeying a Langevin equation

$$\dot{x} = F(x) - \lambda g(x) + \sqrt{\varepsilon}\,\eta\,. \tag{12}$$

The second integral in (11) corresponds to a trajectorial reweighting

$$\exp\Big\{\int_0^{t_{\rm f}} \frac{\lambda}{\varepsilon}\Big[\frac{\lambda}{2}g(x(t))^2 - g(x(t))F(x(t)) - h(x(t))\Big]{\rm d}t\Big\}. \tag{13}$$

Hence, the path integral over $x(t)$ of the full weight ${\rm e}^{-\frac{1}{\varepsilon}S_\lambda}$ is read as the average of the trajectorial reweighting (13) over the realisations of a process $x(t)$ obeying the modified Langevin equation (12). Since the integrand in (13) does not involve any time derivative, one can use the classical Feynman–Kac formula to finally infer the form of the biased operator (10) as follows: in this expression, the Fokker–Planck contribution $-\partial_x\big((F-\lambda g)\,\cdot\,\big) + \frac{1}{2}\varepsilon\partial_x^2\cdot$ corresponds to the modified process (12) and the diagonal part $\frac{\lambda}{\varepsilon}\big(\frac{\lambda}{2}g^2 - gF - h\big)\cdot$ corresponds to the integrand in (13).

Formally, the procedure we have just presented amounts to reinterpreting the biased operator (10), that describes LDFs for combination of current-type and density-type additive observables, into a biased operator for a purely density-type observable (since the integrand of (13) is independent of $\dot{x}$) but for a *different* process, equation (12) instead of equation (1). This procedure is the analogue for diffusions of a similar one that can be devised for Markov jump processes (see e.g. the appendix B of [37]).

From a more physical viewpoint, the exposed procedure shows that even if the biasing resulting from the parameter λ can be partly reabsorbed into the force by changing $F(x)$ into $F(x) - \lambda g(x)$, the Langevin dynamics defined by equation (12) is not equivalent to the biased dynamics defined by the action equation (9), because of the remaining exponential reweighting given in equation (13). This remains true even in the simple case when A is the integrated current (or position of the particle), corresponding to $h(x) = 0$ and $g(x) = 1$. We will explain in the next sections how an effective probability-preserving dynamics, equivalent (in a sense that will be specified) to the biased dynamics defined by the action equation (9), can however be defined using a transformation of the operator $\mathbb{W}_\lambda$.

2.3. Large-deviation principle at large time

We now turn to the study of the large-time and weak-noise scaling behaviour of the distributions at hand. One first remarks from (10) that the biased operator $\mathbb{W}_\lambda$ does not preserve probability (at odds with $\mathbb{W}$, $\langle -|$ is not a left eigenvector of $\mathbb{W}_\lambda$ of eigenvalue 0). In fact, the Perron–Frobenius theorem[8] ensures that the maximal eigenvalue $\varphi_\varepsilon(\lambda)$ of $\mathbb{W}_\lambda$ is real and unique. We now assume that this operator has a gap (this is the case in general if the force is confining or if the space is compact); this ensures that at large time one has

[7] The process described by equation (12) is not related to the effective process in general.
[8] We assume that its conditions of validity are satisfied.

$$\mathrm{e}^{t\mathbb{W}_\lambda} \underset{t\to\infty}{\sim} \mathrm{e}^{t\,\varphi_\varepsilon(\lambda)}\,|R\rangle\langle L| \qquad \text{with} \quad \varphi_\varepsilon(\lambda) = \max\ \mathrm{Sp}\ \mathbb{W}_\lambda \tag{14}$$

where $\langle L|$ and $|R\rangle$ are the corresponding left and right eigenvectors of $\mathbb{W}_\lambda$, normalised as $\langle L|R\rangle = 1$ and $\langle -|R\rangle = 1$. Then, the formal solution $|\hat{P}(t)\rangle = \mathrm{e}^{t\mathbb{W}_\lambda}|P_\mathrm{i}\rangle$ of the evolution equation $\partial_t \hat{P} = \mathbb{W}_\lambda \hat{P}$ implies that

$$\hat{P}(x,\lambda,t) \underset{t\to\infty}{\asymp} \mathrm{e}^{t\,\varphi_\varepsilon(\lambda)}\,R(x). \tag{15}$$

Integrating (7) over x, this implies that, at large time, the moment generating function behaves as

$$\left\langle \mathrm{e}^{-\frac{\lambda}{\varepsilon}A(t)}\right\rangle \underset{t\to\infty}{\asymp} \mathrm{e}^{t\,\varphi_\varepsilon(\lambda)}. \tag{16}$$

This result is an instance of a large deviation function (LDF) exponential scaling [41]: it indicates that the scaled cumulant generating function (SCGF) $\Phi_\varepsilon(\lambda,t)$ defined as

$$\left\langle \mathrm{e}^{-\frac{\lambda}{\varepsilon}A(t)}\right\rangle = \mathrm{e}^{t\,\Phi_\varepsilon(\lambda,t)} \tag{17}$$

goes to a constant at large t: $\lim_{t\to\infty}\Phi_\varepsilon(\lambda,t) = \varphi_\varepsilon(\lambda)$. In other words, all cumulants of the observable $A(t)$ behave linearly in t at large t.

Such LDF scaling can be translated into a large-time behaviour of the distribution of A: integrating (7) over x one gets from (17) that

$$\mathrm{e}^{t\,\Phi_\varepsilon(\lambda,t)} = \int \mathrm{e}^{-\frac{\lambda}{\varepsilon}A}\,P(A,t)\,\mathrm{d}A. \tag{18}$$

Since the lhs behaves exponentially in t at large t, this is compatible with a distribution $P(A \simeq at, t)$ obeying the following scaling

$$P(A\simeq at,t) \underset{t\to\infty}{\asymp} \mathrm{e}^{t\,\pi_\varepsilon(a)}, \tag{19}$$

with φ_ε and π_ε related through

$$\varphi_\varepsilon(\lambda) = \sup_a \left\{\pi_\varepsilon(a) - \frac{\lambda}{\varepsilon}a\right\}. \tag{20}$$

This is an example of large deviation principle, obtained here through a saddle-point analysis of the integral in (18) through the Gärtner–Ellis theorem [45, 46]. It indicates that, in the scaling $A \simeq at$, the distribution of A concentrates exponentially around the most probable value(s) of a, located at the maxima of the function $\pi_\varepsilon(a)$. If $\pi_\varepsilon(a)$ is a concave function of a, then one can invert the Legendre–Fenchel transformation appearing in (20) and obtain

$$\pi_\varepsilon(a) = \inf_\lambda \left\{\varphi_\varepsilon(\lambda) - \frac{\lambda}{\varepsilon}a\right\}. \tag{21}$$

These two Legendre–Fenchel transformations describe the change of ensemble between the microcanonical (fixed a) and canonical (fixed λ) descriptions, at fixed ϵ. The correspondence (21) can be extended at the level of trajectories: under the same convexity hypothesis, the trajectories conditioned to present a given value of $a = A(t_\mathrm{f})/t_\mathrm{f}$ and the trajectories weighted by $\mathrm{e}^{-\frac{1}{\varepsilon}S_\lambda}$ present an asymptotically equivalent distribution as

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$t_{\rm f} \to \infty$, in a sense defined and studied in great depth by Chétrite and Touchette [21, 22], provided that the value of λ is the one which realises the infimum in (21).

2.4. Large-deviation principle in the weak-noise asymptotics $\varepsilon \to 0$

We now consider the opposite order of limits, by keeping the duration $t_{\rm f}$ finite and sending first the noise amplitude to 0. One can use the path-integral representation (8) in order to study the weak-noise asymptotics of the distributions. By a saddle-point evaluation in the $\varepsilon \to 0$ limit, one sees from the definition (17) that, integrating (8) over x, the SCGF behaves as

$$\Phi_\varepsilon(\lambda, t_{\rm f}) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon}\phi(\lambda, t_{\rm f}) \qquad \text{with} \quad \phi(\lambda, t_{\rm f}) = -\frac{1}{t_{\rm f}} \inf_{x(t)} S_\lambda[x(t), t_{\rm f}] \tag{22}$$

where the action $S_\lambda[x(t), t_{\rm f}]$ is defined in (9). The optimisation is performed over trajectories $[x(t)]_{0\leqslant t\leqslant t_{\rm f}}$ of duration $t_{\rm f}$, whose initial position $x(0)$ is sampled according to the initial distribution $P_{\rm i}$ (the simplest case is when $x(0)$ takes a fixed value), and whose final position $x(t_{\rm f})$ is optimised over in the inf of (22). The function $\phi(\lambda, t_{\rm f})$ is a SCGF. Since $\Phi_\varepsilon(\lambda, t_{\rm f})$ converges to $\varphi_\varepsilon(\lambda)$ as $t_{\rm f} \to \infty$ for all ϵ (see (16)), one expects that

$$\lim_{t_{\rm f}\to\infty} \phi(\lambda, t_{\rm f}) = \phi(\lambda) \qquad \text{with} \quad \varphi_\varepsilon(\lambda) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon}\phi(\lambda). \tag{23}$$

Situations where noise-dependent LDFs (here, $\varphi_\varepsilon(\lambda)$) scale as one over the noise strength were for instance considered in [29, 52]. In all, the saddle-point asymptotics provides the following optimisation principle for the SCGF $\phi(\lambda)$ as:

$$\phi(\lambda) = -\lim_{t_{\rm f}\to\infty} \left\{ \frac{1}{t_{\rm f}} \inf_{x(t)} S_\lambda[x(t), t_{\rm f}] \right\}. \tag{24}$$

Hence, the determination of $\phi(\lambda)$ requires the knowledge of the optimal trajectories in the weak-noise limit. This is the topic of the next section.

It is not obvious that the large-time and the weak-noise commute, i.e. that the SCGF $\phi(\lambda)$ given in (24) by first taking $\varepsilon \to 0$ and then $t_{\rm f} \to \infty$ is the same as the $\varepsilon \to 0$ asymptotics $\phi(\lambda) = \lim_{\varepsilon\to 0}[\varepsilon\, \varphi_\varepsilon(\lambda)]$ (see (23)) of the CGF $\varphi_\varepsilon(\lambda)$ obtained from spectral considerations by first taking the $t_{\rm f} \to \infty$ limit, as done in (16). We will show in sections 3 and 4 that these two definitions coincide for periodic systems, i.e. that one can take the large-time and the weak-noise limits in whichever order one prefers.

3. Determination of the SCGF $\phi(\lambda)$ for spatially periodic systems

3.1. Optimal trajectories in the weak-noise limit

We now aim at computing the scaled cumulant generating function $\phi(\lambda)$ by minimising the action $S_\lambda[x(t), t_{\rm f}]$ according to equation (24). The saddle-point equation for the optimal path sustaining a given fluctuation is obtained from the optimisation principle (22) and reads

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$$\ddot{x} - F(x)F'(x) - \lambda h'(x) = 0\,, \tag{25}$$

where the prime denotes a derivative with respect to x. We note that it does not depend on the function $g(x)$ since the term $\dot{x}(t)\,g(x(t))$ in the integrand of the action (9) is a total derivative, but of course the function $g(x)$ still plays a role because it appears in the expression of the action whose value is to be minimised. It represents the conservative dynamics of a particle of unit mass in a potential

$$\mathcal{V}(x) = -\frac{1}{2}F(x)^2 - \lambda h(x). \tag{26}$$

As a result, the energy

$$\mathcal{E}(\dot{x}, x) = \frac{1}{2}\dot{x}^2 + \mathcal{V}(x) \tag{27}$$

is conserved along an optimal trajectory. Let us recall, however, that while optimal trajectories obey a deterministic conservative dynamics in the potential $\mathcal{V}(x)$ given by equation (26), the original dynamics of the problem obeys an overdamped Langevin dynamics, with a deterministic force $F(x)$ that may derive or not from a potential—see equation (1). Note that since $x(t)$ satisfies the second-order differential equation (25), it is uniquely specified in general only when one specifies a set of two parameters. It is convenient to choose for these two parameters the energy $\mathcal{E}$ and the initial position $x_{\rm i}$ (and the sign of the initial velocity, as we explain later).

Importantly, the initial position $x_{\rm i}$ for the optimal trajectory (solution of equation (25)) in the weak-noise asymptotic regime differs from the 'physical' initial condition $x(0)$ considered in the path integral (6) and sampled with $P_{\rm i}$. Indeed, the underlying Langevin dynamics is dissipative, so that the biased distribution $\hat{P}(x, \lambda, t_{\rm f})$ converges to a steady state at large finite time $t_{\rm f}$ after a transient; then, the optimal trajectory that obeys the non-dissipative evolution (25) describes the most probable loci of $\hat{P}(x, \lambda, t_{\rm f})$ In other words, there is a transient regime for the 'physical' initial distribution $P_{\rm i}(x)$ to reach a distribution $\hat{P}(x, \lambda, t_{\rm f})$ that falls into a consistent weak-noise description. As a result, the initial distribution $P_{\rm i}(x)$ becomes irrelevant after this transient regime and can thus be forgotten in the weak-noise evaluation of the path-integral (6), since it does not fall in general in the weak-noise large-deviation regime.

On the other hand, equation (25) has an infinite number of solutions, only one of them being the actual optimal trajectory that minimises the action $S_\lambda[x(t), t_{\rm f}]$. To determine this optimal trajectory, one has to parameterise each solution of equation (25) by the initial position $x_{\rm i}$ and the energy $\mathcal{E}$, but again, $x_{\rm i}$ differs from the 'physical' initial condition $x(0)$.

In the following, we consider finite-size spatially periodic systems, for which $F(x+1) = F(x)$, $h(x+1) = h(x)$ and $g(x+1) = g(x)$ (we took the spatial period as the unit length, without loss of generality). Such periodic systems have been studied in previous works, especially in the weak-noise asymptotics [29], but for a specific additive observable such as the entropy production [24–26] or the current [27–29] and using a numerical approach based on a Fourier–Bloch decomposition [25, 29]. Our aim is to keep the form of the additive observable $A(t_{\rm f})$ generic and the approach analytical for as long as possible in the study of the problem at hand, in order, in particular, to fully characterise dynamical phase transitions that are known to occur [24, 25, 27, 29] in

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this problem but that were not completely understood (the order of such transition for instance remained unclear).

For such spatially periodic systems, the optimal trajectory minimising the action $S_\lambda[x(t), t_f]$ becomes independent of $x(0)$ for large enough time t_f, so that the only relevant parameter to characterise the trajectories in this limit is their energy. To determine the SCGF $\phi(\lambda)$, one has to evaluate the action $S_\lambda[x(t), t_f]$ for any of the optimal trajectories given by equation (25), and to find the optimal trajectory that minimises the action. In practice, this last step consists in minimising the action over the energy of the trajectories. Using equations (9) and (26), the action $S_\lambda[x(t), t_f]$ can be written as

$$S_\lambda[x(t), t_f] = \int_0^{t_f} \mathrm{d}t\, \mathcal{L}(\dot{x}(t), x(t)) \tag{28}$$

with a Lagrangian

$$\mathcal{L}(\dot{x}, x) = \frac{1}{2}\dot{x}^2 - \mathcal{V}(x) + \dot{x}\big(\lambda g(x) - F(x)\big). \tag{29}$$

Note that the last term, proportional to $\dot{x}$, in equation (29) plays no role in equation (25) since it is a total derivative, but it has to be included in the Lagrangian to correctly evaluate (and minimise) the action.

Assuming that the force field $F(x)$ and the function $h(x)$ are bounded, the potential $\mathcal{V}(x)$ is also bounded. We denote as $\mathcal{V}_{\max}$ the maximum value of the potential:

$$\mathcal{V}_{\max} = \max_x \mathcal{V}(x). \tag{30}$$

The value $\mathcal{V}_{\max}$ allows one to classify the optimal trajectories $x^\star(t)$ into periodic and propagative solutions, according to their energy $\mathcal{E}$ (for convenience, we include constant trajectories as a special case of the periodic ones). For $\mathcal{E} < \mathcal{V}_{\max}$, optimal trajectories are confined by the potential $\mathcal{V}(x)$, and are periodic in time. For $\mathcal{E} > \mathcal{V}_{\max}$, the potential no longer confines the optimal trajectories, which are then propagative, with a constant sign of the velocity $\dot{x}$. As a result, the SCGF $\phi(\lambda)$ is obtained by minimising the action over both sets of periodic and propagative optimal trajectories. One can thus write, taking into account the minus sign in equation (24),

$$\phi(\lambda) = \max\left\{\phi_{\mathrm{per}}(\lambda), \phi_{\mathrm{prop}}(\lambda)\right\} \tag{31}$$

where $\phi_{\mathrm{per}}(\lambda)$, $\phi_{\mathrm{prop}}(\lambda)$ are defined by minimising the action over the sets of periodic and propagative trajectories respectively:

$$\phi_{\mathrm{per}}(\lambda) = -\lim_{t_f\to\infty}\left\{\frac{1}{t_f}\inf_{\mathcal{E}<\mathcal{V}_{\max}} S_\lambda[x^\star(t), t_f]\right\}, \tag{32}$$

$$\phi_{\mathrm{prop}}(\lambda) = -\lim_{t_f\to\infty}\left\{\frac{1}{t_f}\inf_{\mathcal{E}>\mathcal{V}_{\max}} S_\lambda[x^\star(t), t_f]\right\}. \tag{33}$$

In the following, we successively evaluate $\phi_{\mathrm{per}}(\lambda)$ and $\phi_{\mathrm{prop}}(\lambda)$.

3.2. Time-periodic trajectories

We start by evaluating $\phi_{\mathrm{per}}(\lambda)$. A particular type of periodic trajectories are the time-independent ones, for which $\dot{x}^\star = 0$ and $x^\star = x_0$, implying $\mathcal{V}'(x_0) = 0$ from equation (25).

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For such trajectories, one has $\mathcal{L}(\dot{x}^\star, x^\star) = -\mathcal{V}(x_0)$, so that minimising the action over time-independent trajectories selects points x_0 that are at the location(s) of the maximum of the potential $\mathcal{V}(x)$; hence:

$$\lim_{t_f\to\infty}\left\{\frac{1}{t_f}\inf_{x(t)=x_0} S_\lambda[x^\star(t),t_f]\right\} = -\mathcal{V}_{\max}. \tag{34}$$

Considering now a generic time-periodic optimal trajectory, the action reads, with $x^\star \equiv x^\star(t)$

$$\frac{1}{t_f}S_\lambda[x^\star(t),t_f] = \underbrace{\frac{1}{2t_f}\int_0^{t_f}\mathrm{d}t(\dot{x}^\star)^2}_{\geqslant 0}\underbrace{-\frac{1}{t_f}\int_0^{t_f}\mathrm{d}t\mathcal{V}(x^\star)}_{\geqslant -\mathcal{V}_{\max}}+\underbrace{\frac{1}{t_f}\int_0^{t_f}\mathrm{d}t\dot{x}^\star\Big(\lambda g(x^\star)-F(x^\star)\Big)}_{\to\ 0\ \text{when}\ t_f\to\infty}. \tag{35}$$

The proof that the last integral in (35) goes to 0 when $t_f \to \infty$ comes from a change of variable from t to x:

$$\frac{1}{t_f}\int_0^{t_f}\mathrm{d}t\,\dot{x}^\star\Big(\lambda g(x^\star)-F(x^\star)\Big) = \frac{1}{t_f}\int_{x^\star(0)}^{x^\star(t_f)}\mathrm{d}x\Big(\lambda g(x)-F(x)\Big) \tag{36}$$

which goes to 0 when $t_f \to \infty$ because $[\lambda g(x) - F(x)]$ is bounded on the finite interval $[x^\star(0), x^\star(t_f)]$. It is tempting to take the $t_f \to \infty$ limit and to conclude from (32) that $\phi_{\mathrm{per}}(\lambda) \leqslant \mathcal{V}_{\max}$ but this would require to exchange the inf and the $t_f \to \infty$ limit in (32), which enters in conflict with our goal since we are interested in how the small-noise and large-time limits commute. To avoid this exchange, one writes from (35) that for any time-periodic optimal trajectory,

$$\frac{1}{t_f}\inf_{\mathcal{E}<\mathcal{V}_{\max}} S_\lambda[x^\star(t),t_f] \geqslant -\mathcal{V}_{\max} - \frac{1}{t_f}\inf_{\mathcal{E}<\mathcal{V}_{\max}}\int_{x^\star(0)}^{x^\star(t_f)}\mathrm{d}x\Big(\lambda g(x)-F(x)\Big) \tag{37}$$

so that taking the $t_f \to \infty$ limit one finds

$$\lim_{t_f\to\infty}\left\{\frac{1}{t_f}\inf_{\mathcal{E}<\mathcal{V}_{\max}} S_\lambda[x^\star(t),t_f]\right\} \geqslant -\mathcal{V}_{\max}, \tag{38}$$

because the integrand on the rhs of (37) is a bounded function on an interval of fixed finite length. From the definition (32), we obtain $\phi_{\mathrm{per}}(\lambda) \leqslant \mathcal{V}_{\max}$. Remarking now from (28)–(29) that this bound is realised for time-independent trajectories $x^\star = x_0$ we conclude that

$$\phi_{\mathrm{per}}(\lambda) = \mathcal{V}_{\max}(\lambda) \tag{39}$$

where the λ-dependence of $\mathcal{V}_{\max}$ has been made explicit. Therefore, for $\mathcal{E} < \mathcal{V}_{\max}$, the optimal trajectories sustaining a given fluctuations are time-independent of the form $x^\star = x_0$, where x_0 are the points maximising the potential $\mathcal{V}(x_0) = \mathcal{V}_{\max}$.

3.3. Propagative trajectories

To evaluate $\phi_{\mathrm{prop}}(\lambda)$, one now has to compute the minimum of the action over all propagative optimal trajectories, i.e. trajectories for which $\mathcal{E} > \mathcal{V}_{\max}$. Then, from energy conservation, one has

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$$\dot{x^\star} = \sigma\sqrt{2\big(\mathcal{E} - \mathcal{V}(x^\star)\big)} \tag{40}$$

where $\sigma = \pm 1$ is the sign of $\dot{x^\star}$ (we recall that the sign of $\dot{x^\star}$ is constant all along propagative trajectories). Propagative trajectories are pseudo-periodic, in the sense that $x^\star(t+T) = x^\star(t) + \sigma$, which may be identified with $x^\star(t)$ due to the spatial periodicity of the system; $T = T(\mathcal{E})$ is the pseudo-period $T(\mathcal{E})$, determined as

$$T = \int_0^T \mathrm{d}t = \int_0^1 \frac{\mathrm{d}x}{\sqrt{2\big(\mathcal{E} - \mathcal{V}(x)\big)}}\,, \tag{41}$$

where we have used equation (40) to change the integration variable from t to x. Using the relation

$$\mathcal{L}(\dot{x^\star}, x^\star) = \mathcal{E} - 2\mathcal{V}(x^\star) + \dot{x^\star}\big(\lambda g(x^\star) - F(x^\star)\big), \tag{42}$$

the action of a propagative optimal trajectory on the time interval $[0, T(\mathcal{E})]$ is, expanding the Lagrangian (29),

$$S_\lambda[x^\star(t), T(\mathcal{E})] = \sigma\int_0^1 \big(\lambda g(x) - F(x)\big)\mathrm{d}x + T(\mathcal{E})\mathcal{E} - \int_0^1 \frac{2\mathcal{V}(x)}{\sqrt{2\big(\mathcal{E} - \mathcal{V}(x)\big)}}\,\mathrm{d}x. \tag{43}$$

To lighten notations, we define

$$B = \int_0^1 \big(\lambda g(x) - F(x)\big)\mathrm{d}x, \qquad R(\mathcal{E}) = \int_0^1 \frac{2\mathcal{V}(x)}{\sqrt{2\big(\mathcal{E} - \mathcal{V}(x)\big)}}\,\mathrm{d}x. \tag{44}$$

Note that the term $F(x)$ in the integral defining B gives no contribution when the force $F(x)$ derives from a potential.

In the large-time limit, the value of the action over every interval $[nT(\mathcal{E}), (n+1)T(\mathcal{E})]$ is the same (by periodicity of the optimal trajectory). Furthermore, the optimal trajectory dependence on the initial value x_0 is now replaced by a pseudo-periodic boundary condition of the form $x(1) = x(0) + \sigma$. Hence the $\phi_{\mathrm{prop}}(\lambda)$ defined in equation (33) is equal to:

$$\phi_{\mathrm{prop}}(\lambda) = -\lim_{n\to\infty} \inf_{\mathcal{E}>\mathcal{V}_{\max}, \sigma=\pm 1} \left\{ \frac{1}{nT(\mathcal{E})} \int_0^{nT(\mathcal{E})} \mathcal{L}(\dot{x}^\star, x^\star)\,\mathrm{d}t \right\} \tag{45}$$

$$= -\inf_{\mathcal{E}>\mathcal{V}_{\max}, \sigma=\pm 1} \left\{ \frac{1}{T(\mathcal{E})} \int_0^{T(\mathcal{E})} \mathcal{L}(\dot{x}^\star, x^\star)\,\mathrm{d}t \right\} \tag{46}$$

$$= -\inf_{\mathcal{E}>\mathcal{V}_{\max}, \sigma=\pm 1} \left\{ \mathcal{E} + \frac{\sigma B - R(\mathcal{E})}{T(\mathcal{E})} \right\} \tag{47}$$

where in (45)–(46) the optimal trajectory $x^\star(t)$ is the propagative solution of the saddle-point equation, with an energy $\mathcal{E}$ and a pseudo-period $T(\mathcal{E})$ that depends on $\mathcal{E}$, as inferred from (41). Determining $\phi_{\mathrm{prop}}(\lambda)$ thus amounts to finding, for both $\sigma = \pm 1$, the infimum of the function

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$$\Psi_\sigma(\mathcal{E}) = \mathcal{E} + \frac{\sigma B - R(\mathcal{E})}{T(\mathcal{E})}. \tag{48}$$

The function $\Psi_\sigma(\mathcal{E})$ is defined over the interval $(\mathcal{V}_{\max}, +\infty)$. When $\mathcal{E} \to \mathcal{V}_{\max}$, both $R(\mathcal{E})$ and $T(\mathcal{E})$ diverge to infinity (assuming $\mathcal{V}(x)$ is regular close to $\mathcal{V}_{\max}$), but their ratio $R(\mathcal{E})/T(\mathcal{E}) \to 2\mathcal{V}_{\max}$, so that $\Psi_\sigma(\mathcal{E}) \to -\mathcal{V}_{\max}$. In the opposite limit $\mathcal{E} \to \infty$, $T(\mathcal{E}) \sim R(\mathcal{E}) \sim 1/\sqrt{\mathcal{E}}$, yielding $\Psi_\sigma(\mathcal{E}) \to +\infty$. Consequently, if $\Psi_\sigma(\mathcal{E})$ has no minimum for $\mathcal{E} \in (\mathcal{V}_{\max}, +\infty)$, one has:

$$\inf_{\mathcal{E} > \mathcal{V}_{\max}} \Psi_\sigma(\mathcal{E}) = -\mathcal{V}_{\max}. \tag{49}$$

We now proceed to determine if $\Psi_\sigma(\mathcal{E})$ has a minimum $\mathcal{E}^*_\sigma$, satisfying $\Psi'_\sigma(\mathcal{E}^*_\sigma) = 0$. The derivative $\Psi'_\sigma(\mathcal{E})$ reads

$$\Psi'_\sigma(\mathcal{E}) = \frac{1}{T(\mathcal{E})^2}\Big[T(\mathcal{E})^2 - R'(\mathcal{E})T(\mathcal{E}) + R(\mathcal{E})T'(\mathcal{E}) - \sigma B T'(\mathcal{E})\Big]. \tag{50}$$

From the definition (44) of $R(\mathcal{E})$, one finds that

$$R'(\mathcal{E}) = T(\mathcal{E}) + 2\mathcal{E}T'(\mathcal{E}) \tag{51}$$

so that $\Psi'_\sigma(\mathcal{E})$ can be rewritten as

$$\Psi'_\sigma(\mathcal{E}) = \frac{T'(\mathcal{E})}{T(\mathcal{E})^2}\Big[R(\mathcal{E}) - 2\mathcal{E}T(\mathcal{E}) - \sigma B\Big]. \tag{52}$$

Since $T'(\mathcal{E}) \neq 0$ for all $\mathcal{E}$, the condition $\Psi'_\sigma(\mathcal{E}^*_\sigma) = 0$ is equivalent to

$$R(\mathcal{E}^*_\sigma) - 2\mathcal{E}^*_\sigma T(\mathcal{E}^*_\sigma) - \sigma B = 0 \tag{53}$$

which determines $\mathcal{E}^*_\sigma$. If a solution $\mathcal{E}^*_\sigma$ exists, one has from equations (48) and (53)

$$\Psi_\sigma(\mathcal{E}^*_\sigma) = \mathcal{E}^*_\sigma + \frac{\sigma B - R(\mathcal{E}^*_\sigma)}{T(\mathcal{E}^*_\sigma)} = -\mathcal{E}^*_\sigma. \tag{54}$$

Using equations (51) and (53), one can show that the second derivative $\Psi''_\sigma(\mathcal{E}^*_\sigma)$ takes the simple form

$$\Psi''_\sigma(\mathcal{E}^*_\sigma) = -\frac{T'(\mathcal{E}^*_\sigma)}{T(\mathcal{E}^*_\sigma)} = \frac{\int_0^1 \big(\mathcal{E}^*_\sigma - \mathcal{V}(x)\big)^{-3/2}\,\mathrm{d}x}{2\int_0^1 \big(\mathcal{E}^*_\sigma - \mathcal{V}(x)\big)^{-1/2}\,\mathrm{d}x} > 0, \tag{55}$$

so that $\mathcal{E}^*_\sigma$ is a local minimum. The fact that it is a global minimum comes from a unicity argument, which goes as follows. Equation (53) can be rewritten using equations (41) and (44) as

$$\int_0^1 \sqrt{2\big(\mathcal{E}^*_\sigma - \mathcal{V}(x)\big)}\,\mathrm{d}x = -\sigma B. \tag{56}$$

The integral on the lhs of equation (56) spans the interval $\big(\int_0^1 \sqrt{2(\mathcal{V}_{\max} - \mathcal{V}(x))}\,\mathrm{d}x, +\infty\big)$ as a function of $\mathcal{E}^*_\sigma$. Hence a solution $\mathcal{E}^*_\sigma$ exists if

$$-\sigma B > \int_0^1 \sqrt{2\big(\mathcal{V}_{\max} - \mathcal{V}(x)\big)}\,\mathrm{d}x \tag{57}$$

where we recall that B is defined in equation (44). Since $\int_0^1 \sqrt{2(\mathcal{E}-\mathcal{V}(x))}\,\mathrm{d}x$ is an increasing function of $\mathcal{E}$, the solution $\mathcal{E}_\sigma^*$ is unique if it exists. Hence the function $\Psi_\sigma(\mathcal{E})$ has at most one stationary point, so that its local minimum $\mathcal{E}_\sigma^*$ is, if it exists, a global minimum. In addition, if $\mathcal{E}_\sigma^*$ exists, then $\mathcal{E}_{-\sigma}^*$ does not exist, since equation (57) cannot be simultaneously satisfied for σ and $-\sigma$. This implies that $\mathcal{E}_\sigma^*$ can exist only for

$$\sigma = -\mathrm{sign}(B), \tag{58}$$

and equation (57) can be rewritten as

$$|B| > \int_0^1 \sqrt{2\big(\mathcal{V}_{\max} - \mathcal{V}(x)\big)}\,\mathrm{d}x. \tag{59}$$

If equation (59) is satisfied, one has for $\sigma = -\mathrm{sign}(B)$ that

$$\inf_{\mathcal{E}>\mathcal{V}_{\max}} \Psi_\sigma(\mathcal{E}) = -\mathcal{E}_\sigma^*, \quad \text{and} \quad \inf_{\mathcal{E}>\mathcal{V}_{\max}} \Psi_{-\sigma}(\mathcal{E}) = -\mathcal{V}_{\max}. \tag{60}$$

As a result, equation (47) implies

$$\phi_{\mathrm{prop}}(\lambda) = -\min\{-\mathcal{E}_\sigma^*, -\mathcal{V}_{\max}\} = \max\{\mathcal{E}_\sigma^*, \mathcal{V}_{\max}\} = \mathcal{E}_\sigma^* \tag{61}$$

(with $\sigma = -\mathrm{sign}(B)$) if equation (59) holds.

In the opposite case, if equation (59) is not satisfied,

$$\inf_{\mathcal{E}>\mathcal{V}_{\max}} \Psi_\sigma(\mathcal{E}) = \inf_{\mathcal{E}>\mathcal{V}_{\max}} \Psi_{-\sigma}(\mathcal{E}) = -\mathcal{V}_{\max}, \tag{62}$$

so that

$$\phi_{\mathrm{prop}}(\lambda) = \mathcal{V}_{\max}. \tag{63}$$

Since from equation (39) $\phi_{\mathrm{per}}(\lambda) = \mathcal{V}_{\max} \leqslant \phi_{\mathrm{prop}}(\lambda)$, equation (31) implies that for all λ

$$\phi(\lambda) = \phi_{\mathrm{prop}}(\lambda). \tag{64}$$

To summarise, we have shown that when a propagative optimal solution exists, it is unique and the value of the corresponding SCGF is given by the *energy* $\mathcal{E}_\sigma^*$ of such trajectory; otherwise, the value of the SCGF is given by the maximum value $\mathcal{V}_{\max}$ of the potential $\mathcal{V}(x)$. This result is surprisingly simple in view of the complicated optimisation problem one is initially facing, and it is key to the explicit determination of the SCGF. We emphasise that although the search of optimal trajectories (coming as a standard consequence of the weak-noise approach) is formulated in the framework of Lagrangian mechanics, the result we have obtained goes one step beyond: in Lagrangian mechanics indeed the conserved energy $\mathcal{E}$ of trajectories is given and fixed, while in our problem of interest the energy itself has to be optimised. It is precisely the optimal energy $\mathcal{E}^*$ that benefits of the unexpected property that the action of its corresponding optimal trajectory becomes equal to $\mathcal{E}^*$—a non-trivial fact, as we have shown. Such result could be of interest in different contexts where Lagrangian mechanics is used to solve optimisation problems. We note last that our result is not directly related to the optimisation principle put forward by Nemoto and Sasa in [27, 28] (since these works are not based on a weak-noise framework).

In conclusion, the evaluation of $\phi(\lambda)$ generically goes as follows. Note as a starting point that the criterion given by equation (59) for the existence of an optimal

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propagative solution can be interpreted as a condition on λ; using explicit notations, one has that if the condition:

$$\left|\lambda \int_0^1 g(x)\,\mathrm{d}x - \int_0^1 F(x)\,\mathrm{d}x\right| > \int_0^1 \sqrt{2\mathcal{V}_{\max}(\lambda) + F(x)^2 + 2\lambda h(x)}\,\mathrm{d}x \qquad (65)$$

is satisfied, then the optimal trajectory is propagative. For each value of λ, one checks whether equation (65) is satisfied. If it holds, one determines $\mathcal{E}_\sigma^*$ by solving equation (53) with $\sigma = -\mathrm{sign}(B)$, i.e.

$$R(\mathcal{E}_\sigma^*) - 2\mathcal{E}_\sigma^* T(\mathcal{E}_\sigma^*) + |B| = 0, \qquad (66)$$

leading to $\phi(\lambda) = \mathcal{E}_\sigma^*(\lambda)$. If equation (65) is not satisfied, then $\phi(\lambda) = \mathcal{V}_{\max}(\lambda)$. This result allows one to determine the existence of possible dynamical phase transitions in the fluctuations of the additive observable $A(t_\mathrm{f})$. When varying λ, one can indeed jump from a situation where the optimal trajectory is time-independent (when equation (65) is not satisfied) to a situation where the optimal trajectory is time-dependent. Such a transition between two classes of optimal trajectories is illustrated in figure 1 and corresponds to a breaking of the 'additivity principle' [49]).

For a density-type observable $A(t_\mathrm{f})$ ($g(x) = 0$) in the presence of a conservative force $F(x) = -U'(x)$, the lhs of equation (65) is equal to 0, so that equation (65) is never satisfied. It follows that $\phi(\lambda) = \mathcal{V}_{\max}(\lambda)$ for all λ, meaning that the optimal trajectory is always time-independent in this case (in other words, there is no breaking of the additivity principle). This however does not forbid dynamical phase transitions since, as seen from the expression (26) of $\mathcal{V}(x)$ the 'tilting' contribution $-\lambda h(x)$ can make the location of the maximum of $\mathcal{V}(x)$ switch from one position to another, if for instance $F(x)$ presents more than one equilibrium point. Such a situation occurs for instance in the large deviation of additive observables in driven diffusive systems [53].

We discuss below the determination of $\phi(\lambda)$ in the case of current-type additive observable, which generically leads to a phase transition between stationary and non-stationary trajectories—a common phenomenon in periodic systems in general [10, 29, 33, 54].

3.4. Determination of $\phi(\lambda)$ for current-type additive observable ($h(x) = 0$) and conservative force $F(x)$

Considering a current-type additive observable (corresponding to $h(x) = 0$) as well as a conservative force $F(x) = -U'(x)$, equation (65) simplifies to

$$|\lambda| > \lambda_\mathrm{c} \equiv \frac{\int_0^1 |F(x)|\,\mathrm{d}x}{\int_0^1 g(x)\,\mathrm{d}x} \qquad (67)$$

where we have assumed that $\int_0^1 g(x)\,\mathrm{d}x > 0$ (the case $\int_0^1 g(x)\,\mathrm{d}x < 0$ is treated in the same way), and used the fact that $\mathcal{V}_{\max} = 0$ when $h(x) = 0$ (as inferred from (26) and (30)). For $|\lambda| > \lambda_\mathrm{c}$, $\phi(\lambda)$ is solution of the equation

$$\int_0^1 \sqrt{2\phi(\lambda) + F(x)^2}\,\mathrm{d}x = |\lambda| \int_0^1 g(x)\,\mathrm{d}x \qquad (68)$$

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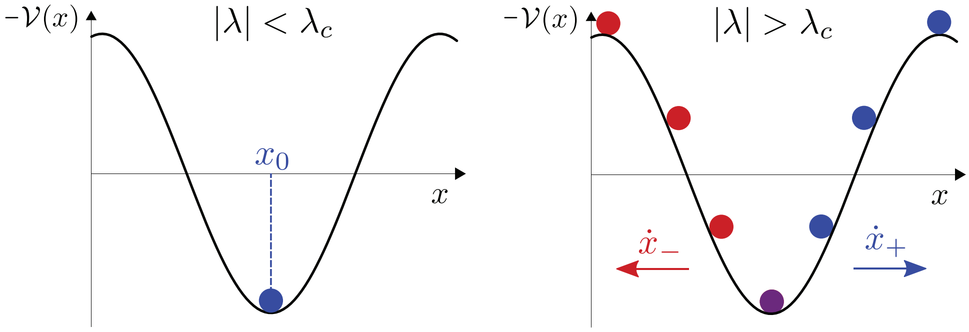

Figure 1. The different classes of optimal trajectories of interest: On the left, stationary ones, with x_0 the location of the maximum of $\mathcal{V}(x)$, defined in equation (26); On the right, propagative ones, either increasing or decreasing in time. Periodic trajectories which oscillate around x_0 without being propagative have a larger action than the stationary one in x_0, as shown in section 3.2. The criterion for the existence of propagative trajectories is given by equation (65).

while for $|\lambda| < \lambda_c$, $\phi(\lambda) = \mathcal{V}_{\max} = 0$. Consequently, λ_c is the critical value at which the dynamical phase transition takes place. Note that $\phi(\lambda)$ is an even function of λ for current-type additive observable in a system with a conservative force, due to time-reversal invariance.

The singular behaviour of the SCGF close to the transition depends both on local and global properties of the force $F(x)$. If one naively expands for small $\phi(\lambda)$

$$\sqrt{F(x)^2 + 2\phi(\lambda)} = |F(x)| + \frac{\phi(\lambda)}{|F(x)|} + O\big(\phi(\lambda)^2\big) \tag{69}$$

and integrates over $0 < x < 1$ in order to obtain an expansion of equation (68) for $\phi(\lambda)$, one finds a divergent integral $\int_0^1 \mathrm{d}x/|F(x)|$ (if there is a fixed point $F(x_0) = 0$ with $F'(x_0) \neq 0$, which is the case in general). The expansion at small $\phi(\lambda)$ is thus ill-defined. The logarithmic divergence of $\int_0^1 \mathrm{d}x/|F(x)|$ suggests a behaviour $\phi(\lambda) \sim (\lambda - \lambda_c)/|\log(\lambda - \lambda_c)|$ but the situation is better understood on a specific example first.

As an explicit example, we consider the case $F(x) = \sin(2\pi x)$, $h(x) = 0$ and $g(x) = 1$, where the force $F(x)$ is conservative, and the observable $A(t)$ is the integrated current (or the position at time t, counted with the number of turns). Such model and additive observable have been previously considered in [25–27, 29]. The SCGF $\phi(\lambda)$ was numerically evaluated in [27] at finite noise (the transition that appears in the weak-noise limit was not considered). In [25, 26] the weak-noise limit is considered and the singularity of the SCGF is described as a 'kink', with no elucidation of its nature. In [29], numerical results in the weak but non-zero noise asymptotics are obtained, and described as suggesting a first-order transition with a cusp in the SCGF (compatible with the results of random-walk approximation of the problem), but the authors write that 'the precise shape of the rate function around the cusp is yet to be determined analytically'. Below, we elucidate the precise form of the dynamical phase transition that appears in the weak-noise asymptotics, showing that it is neither first nor second order, but continuous and intermediate between these two cases (it presents an essential singularity). For the location of the transition, equation (67) straightforwardly leads to

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$$\lambda_{\mathrm{c}} = \frac{2}{\pi}. \tag{70}$$

This result was already obtained in [29]. For $|\lambda| > \lambda_{\mathrm{c}}$, the SCGF $\phi(\lambda)$ is solution in $\mathcal{E}_\lambda$ of the equation

$$\lambda = \begin{cases} -\Lambda(\mathcal{E}_\lambda) & \text{for } \lambda < -\lambda_{\mathrm{c}} \\ \Lambda(\mathcal{E}_\lambda) & \text{for } \lambda > \lambda_{\mathrm{c}}, \end{cases} \tag{71}$$

with

$$\Lambda(\mathcal{E}) \equiv \frac{2}{\pi}\sqrt{2\mathcal{E}}\ \mathrm{E}\left(-\frac{1}{2\mathcal{E}}\right), \tag{72}$$

where $\mathrm{E}(\cdot)$ is the complete elliptic integral of the second kind (taking the definition used by Abramowitz and Stegun [55]). One thus obtains

$$\phi(\lambda) = \begin{cases} 0 & |\lambda| \leqslant \lambda_{\mathrm{c}}, \\ \Lambda^{-1}(|\lambda|) & |\lambda| \geqslant \lambda_{\mathrm{c}}. \end{cases} \tag{73}$$

For $|\lambda| < \lambda_{\mathrm{c}}$, the SCGF $\phi(\lambda)$ and its associated optimal profiles are flat: one needs to consider large enough deviations of the current in order to actually observe a travelling trajectory. In order to check the existence of the transition and the form of the SCGF, we evaluated $\phi(\lambda)$ from the maximal eigenvalue of the deformed operator (10) of a lattice version of the dynamics (at small temperature and for a large number of sites). Results are in good agreement with the present weak-noise approach (see figure 2).

For the expansion close to the transition points, one finds for $\lambda = \lambda_{\mathrm{c}} + \delta\lambda$ with $\delta\lambda > 0$

$$\phi(\lambda_{\mathrm{c}} + \delta\lambda) = \frac{\pi\,\delta\lambda}{|\ln\delta\lambda|} + o\left(\frac{\delta\lambda}{|\ln\delta\lambda|}\right). \tag{74}$$

This leads for the average velocity $\overline{v}$ of the particle (or, equivalently, the average current),

$$\overline{v}(\lambda_{\mathrm{c}} + \delta\lambda) = -\phi'(\lambda_{\mathrm{c}} + \delta\lambda) = -\frac{\pi}{|\ln\delta\lambda|} + o\left(\frac{1}{|\ln\delta\lambda|}\right). \tag{75}$$

As a result, the dynamical phase transition at λ_{c} is formally continuous since $\overline{v}(\lambda) \to 0$ when $\lambda \to \lambda_{\mathrm{c}}$. However, for all practical purposes, the transition appears discontinuous as the convergence to zero is extremely slow. The higher-order derivatives $\phi^{(n)}(\lambda)$ diverge to $(-1)_n\infty$ as $\lambda \to \lambda_{\mathrm{c}}^+$ for $n \geqslant 2$, indicating an essential singularity of the SCGF in λ_{c}. This result, which is new to our knowledge and highly non-trivial, is to be contrasted with the standard depinning transition of a particle in a 'tilted' potential (i.e. a particle subjected to a conservative force plus a uniform non-conservative driving force). For a regular potential, the depinning transition is continuous with a critical exponent 1/2 [56]. The fact that the transition observed in the λ-biased dynamics is of a different nature shows that biasing the dynamics with λ does not only add a non-conservative uniform driving force to the original dynamics, but rather modifies the original dynamics in a more complex way. We describe in the next section how the λ-biased dynamics can be mapped onto a non-trivial effective driven process, which will allow us to better

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understand why the standard behaviour of the depinning transition of a particle in a potential is not recovered (see also [29] for a complementary approach).

4. Effective non-equilibrium dynamics of the conditioned equilibrium system

As we discussed in section 2.3, the biased dynamics is governed by the deformed Fokker–Planck operator $\mathbb{W}_\lambda$ defined in (10), which does not preserve probability. This observation is at the basis of population dynamics algorithms [57–59] that allow one to study rare trajectories and to evaluate numerically the CGF by representing the probability loss or gain through selection rules between copies of the system, in the spirit of Quantum Monte Carlo algorithms [60, 61] (see e.g. [62] for a review).

In fact, as shown recently in [18, 19] and in [20] (inspired by [63, 64]), there exists a change of basis, based on the explicit knowledge of the left eigenvector of $\mathbb{W}_\lambda$, that allows one to render the dynamics described by $\mathbb{W}_\lambda$ *probability-preserving*, up to a global normalisation. As studied in great depth in [21, 22, 65], this defines an 'auxiliary' or 'effective' dynamics $\mathbb{W}_\lambda^{\text{eff}}$ which is asymptotically equivalent to the biased dynamics described by $\mathbb{W}_\lambda$, if normalised appropriately (see section 4.4 for details). From a mathematical point of view, this construction is based on a generalised Doob's h-transform [22]. The interest of this effective dynamics is that it provides a physical (probability-preserving) dynamics whose typical trajectories are equivalent to the rare trajectories of the original dynamics (1). Such effective dynamics can be defined for Langevin processes or for jump processes. Explicit examples of such dynamics have been determined in exclusion processes [18, 19, 66, 67], in zero-range processes [68–70], in the current large deviation of Langevin dynamics [29] or in open quantum systems [71, 72]. They illustrate in general that the effective forces governing the dynamics described by $\mathbb{W}_\lambda^{\text{eff}}$ modify the original dynamics (1) on a global scale.

In this section, we recall how to identify the effective process as a Langevin dynamics with a force $F_\lambda^{\text{eff}}(x)$ that defines a λ-modified probability-preserving dynamics [21, 22]. We then show that the determination of $F_\lambda^{\text{eff}}(x)$ can be done in a rather explicit way in the weak-noise limit, without having to determine explicitly the left eigenvector. We also explain how the determination of the effective process allows one to show that the small-noise and large-time limits can be exchanged in periodic systems for our LDF problem.

4.1. Derivation of the effective force

One defines $\langle L|$ as the left eigenvector[9] of $\mathbb{W}_\lambda$ associated with the maximal eigenvalue $\varphi_\varepsilon(\lambda)$. Following [18–20], one introduces a diagonal operator $\hat{L}$ whose elements are the components of $\langle L|$. Then, the definition $\langle L|\mathbb{W} = \varphi_\varepsilon(\lambda)\langle L|$ of the left eigenvector implies that

[9] It is unique up to a multiplicative constant, since we have assumed that the conditions of validity of the Perron–Frobenius theorem are satisfied. Note also that this implies that all components of $\langle -|$ can be chosen to be strictly positive.

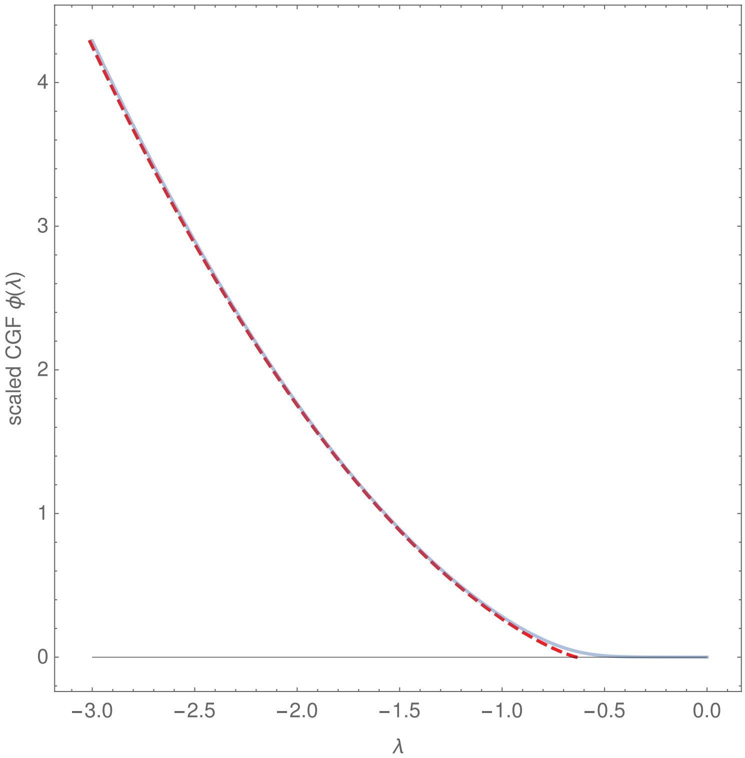

Figure 2. An example SCGF $\phi(\lambda)$, for $F(x) = \sin(2\pi x)$ and $h(x) = 0$ and $g(x) = 1$. Comparison of the evaluation of the SCGF $\phi(\lambda)$ between the weak-noise approach (deduced from (71)–(73), dashed red line) and the maximal eigenvalue of the deformed operator (10) of a lattice version of the dynamics (translucent blue line; 128 sites, $\varepsilon = 0.075$). In the negative regime of λ, the transition occurs at $\lambda = -\lambda_c = -\frac{2}{\pi} \simeq -0.637$.

$$\langle - | \mathbb{W}_\lambda^{\text{eff}} = 0 \quad \text{with} \quad \mathbb{W}_\lambda^{\text{eff}} = \hat{L} \mathbb{W}_\lambda \hat{L}^{-1} - \varphi_\varepsilon(\lambda) \cdot . \tag{76}$$

One reads from this relation that the operator $\mathbb{W}_\lambda^{\text{eff}}$ is probability-preserving and describes a *bona fide* dynamics (with positive transition rates). Since $\mathbb{W}_\lambda^{\text{eff}}$ is obtained from $\mathbb{W}_\lambda$ by a mere shift and a change of basis, it has to present the same physical content as $\mathbb{W}_\lambda$—but in a sense that has to be specified. We refer to [20–22] for thorough studies of such a correspondence and to section 4.4 for a self-contained presentation.

Let us now fix $\varepsilon > 0$ (not necessarily small) and send t_f to infinity before taking the weak-noise limit. The Perron–Frobenius theorem ensures that the components of the eigenvectors associated with the large eigenvalue can be taken to be strictly positive, which allows one to introduce a function $\tilde{U}(x)$ such that the left eigenvector reads $L(x) = \mathrm{e}^{-\frac{1}{\varepsilon}\tilde{U}(x)}$. Note that this relation is simply a definition of $\tilde{U}(x)$, which may depend on ε at this stage. However, one expects[10] that in the weak-noise limit $\varepsilon \to 0$, $\tilde{U}(x)$ becomes independent of ε.

[10] This can be formally shown for instance by expanding $\tilde{U}(x)$ in a power series as $\tilde{U}(x) = \tilde{U}_0(x) + \varepsilon \tilde{U}_1(x) + \ldots$ following the standard WKB procedure [23].

One finds by direct computation that

$$\begin{aligned}\mathbb{W}_\lambda^{\text{eff}} P(x) =& \frac{1}{2}\varepsilon P''(x) + \big[-F(x)+\lambda g(x)+\tilde{U}'(x)\big]P'(x)\\ &+\frac{1}{2\varepsilon}\Big[\lambda^2 g(x)^2 - 2\lambda h(x) - 2\varepsilon\big(\varphi_\varepsilon(\lambda)+F'(x)-\lambda g'(x)\big)\\ &\qquad - 2F(x)\tilde{U}'(x)+\tilde{U}'(x)^2+2\lambda g(x)(\tilde{U}'(x)-F(x))+\varepsilon\tilde{U}''(x)\Big]P(x). \end{aligned} \tag{77}$$

Using the eigenvector equation for $L(x)$

$$\frac{1}{2}\varepsilon L''(x)+\big[F(x)-\lambda g(x)\big]L'(x)+\frac{\lambda}{\varepsilon}\left[\frac{1}{2}\lambda g(x)^2-h(x)-F(x)g(x)\right]L(x)=\varphi_\varepsilon(\lambda)L(x) \tag{78}$$

which can be rewritten in terms of $\tilde{U}(x)$ as

$$-\frac{1}{2}\tilde{U}''(x)+\frac{1}{\varepsilon}\left[\frac{1}{2}\big(\lambda g(x)+\tilde{U}'(x)\big)\big(\lambda g(x)+\tilde{U}'(x)-2F(x)\big)-\lambda h(x)\right]=\varphi_\varepsilon(\lambda), \tag{79}$$

one eliminates $\varphi_\varepsilon(\lambda)$ in (77) and one finds that $\mathbb{W}_\lambda^{\text{eff}}$ indeed takes the form of a probability-preserving Fokker–Planck evolution operator [21, 22]:

$$\mathbb{W}_\lambda^{\text{eff}}\cdot = -\partial_x\Big[\big(F(x)-\lambda g(x)-\tilde{U}'(x)\big)\cdot\Big]+\frac{1}{2}\varepsilon\partial_x^2\cdot\ . \tag{80}$$

It describes the evolution of a particle subjected to a force $F^{\text{eff}}(x)=F(x)-\lambda g(x)-\tilde{U}'(x)$. We note that the contribution $h(x)$ to the additive observable A defined in (2) does not appear explicitly in (80) but is still present implicitly through the potential $\tilde{U}(x)$ defined from the left eigenvector $L(x)$.

4.2. Effective dynamics in the weak-noise limit

Noting that $\varphi_\varepsilon(\lambda)\sim\frac{1}{\varepsilon}\phi(\lambda)$ in the weak-noise limit $\varepsilon\to 0$, and assuming that $\tilde{U}$ becomes independent of ε in this limit (as is usually the case in this WKB procedure [23]), the differential equation (79) for $\tilde{U}(x)$ becomes an ordinary, quadratic equation for $\tilde{U}'(x)$,

$$\frac{1}{2}\big(\lambda g(x)+\tilde{U}'(x)\big)\big(\lambda g(x)+\tilde{U}'(x)-2F(x)\big)-\lambda h(x)=\phi(\lambda), \tag{81}$$

whose solution reads

$$\tilde{U}'(x)=F(x)-\lambda g(x)-\sigma\sqrt{F(x)^2+2\lambda h(x)+2\phi(\lambda)}, \tag{82}$$

where $\sigma=\pm 1$ is an unknown sign that will be determined later on. Hence, the knowledge of $\phi(\lambda)$ allows for the determination of $\tilde{U}'(x)$, provided one is able to select the correct sign in equation (82). This can be done by evaluating the effective force $F^{\text{eff}}(x)$. Inserting equation (82) in the generic expression (80) of the effective Fokker–Planck operator, one finds

$$\mathbb{W}_\lambda^{\text{eff}}\cdot = -\partial_x\Big[\sigma\sqrt{F(x)^2+2\lambda h(x)+2\phi(\lambda)}\ \cdot\Big]+\frac{1}{2}\varepsilon\partial_x^2\cdot\ . \tag{83}$$

It corresponds to the evolution of a particle subjected to an effective force

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$$F^{\text{eff}}(x) = \sigma\sqrt{F(x)^2 + 2\lambda h(x) + 2\phi(\lambda)}. \tag{84}$$

The two possible signs correspond to the two possible cases of sections 3.3 and 3.4 when the optimal trajectory is either increasing or decreasing in time. We will see below that σ is given by $\sigma = -\text{sign}(B)$, consistently with the results of section 3.3.

We have thus shown that in the weak-noise asymptotics, the explicit knowledge of the complete left eigenvector $\langle L|$ is not required in order to determine the effective force $F^{\text{eff}}(x)$: one only needs to know the SCGF $\phi(\lambda)$. Interestingly, for periodic systems, equation (82) also provides a way to determine $\phi(\lambda)$, without using the optimisation procedure described in section 3. In a periodic system, $\tilde{U}(x)$ is a periodic function of period 1, so that $\int_0^1 \tilde{U}'(x)\,\mathrm{d}x = 0$. From equation (82), we thus have

$$\int_0^1 \mathrm{d}x\,\left(F(x) - \lambda g(x) - \sigma\sqrt{F(x)^2 + 2\lambda h(x) + 2\phi(\lambda)}\right) = 0 \tag{85}$$

and one recovers equation (53), given the definitions (44) and (26) of the parameter B, the function R and the potential $\mathcal{V}(x)$, as well as the identification of $\phi(\lambda)$ with $\mathcal{E}_\sigma^*$ when equation (53) has a solution—see equations (61) and (64). Following the same reasoning as the one that leads to equation (58), we recover that $\sigma = -\text{sign}(B)$.

Note that recovering the same result as in section 3 is non-trivial, because here we have made no optimisation of the action at finite time t_{f}, but rather taken the infinite-time limit from the outset, by using first a spectral analysis (which yielded the eigenvector equation (78)) and then a weak-noise expansion to go from (79) to (81). In other words, we have exchanged the order of the large-time and the weak-noise limit. It is interesting to see, as we previously mentioned, that both orders of limits yield the same result. Let us emphasise that this result strongly relies on the Perron–Frobenius theorem, which states that the eigenvector $\langle L|$ associated with the largest eigenvalue of $\mathbb{W}_\lambda$ only has strictly positive components (up to a sign convention) so that it can be written in an exponential form $L(x) = \mathrm{e}^{-\frac{1}{\varepsilon}\widetilde{U}(x)}$ (with real $\widetilde{U}(x)$), while other eigenvectors do not have all components of the same sign, and can thus not be written in such an exponential form. Looking for an eigenvector in an exponential form thus automatically selects the eigenvector associated with the largest eigenvalue thanks to the Perron–Frobenius theorem, without any explicit optimisation procedure.

For a conservative force $F(x)$ and current-type additive observables (i.e. $h(x) = 0$), the condition $\sigma = -\text{sign}(B)$ boils down (if $g(x) > 0$) to $\sigma = -\text{sign}(\lambda)$, leading to an effective force

$$F^{\text{eff}}(x) = -\text{sign}(\lambda)\,\sqrt{F(x)^2 + 2\phi(\lambda)} \qquad (|\lambda| > \lambda_{\text{c}}), \tag{86}$$

$$F^{\text{eff}}(x) = -\text{sign}(\lambda)\,|F(x)|(|\lambda| \leqslant \lambda_{\text{c}}). \tag{87}$$

Several comments are in order here. First, the effective force $F^{\text{eff}}(x)$ differs from the λ-dependent force appearing in the modified process defined by the Langevin equation (12). Second, $F^{\text{eff}}(x)$ can be decomposed into a uniform non-conservative part

$$f_{\text{eff}} = \int_0^1 F^{\text{eff}}(x)\,\mathrm{d}x = \int_0^1 F(x)\,\mathrm{d}x - \lambda\int_0^1 g(x)\,\mathrm{d}x \tag{88}$$

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(where the last equality results from equation (85)) and a space-dependent conservative part

$$-U'_{\rm eff} = F^{\rm eff}(x) - f_{\rm eff}. \tag{89}$$

Note that the integrals in equation (88) should be understood as spatial averages (we recall that the length of the system is chosen as $L = 1$). In the specific case when the observable A is the current (i.e. $g(x) = 1$ and $h(x) = 0$) and $F(x)$ is a conservative force, one recovers that $f_{\rm eff} = -\lambda$. Yet, the conservative part $-U'_{\rm eff}$ does not reduce to the original force $F(x)$. This can be seen explicitly by computing perturbatively the effective force $F^{\rm eff}(x)$ in the large-λ limit, yielding

$$F^{\rm eff}(x) \underset{\lambda\to\infty}{=} -\lambda - \frac{1}{2\lambda}\left(F(x)^2 - \int_0^1 F(x')^2\,{\rm d}x'\right) + o\left(\frac{1}{\lambda}\right). \tag{90}$$

From this last equation, the decomposition of the effective force $F^{\rm eff}(x)$ into a uniform non-conservative force $f_{\rm eff} = -\lambda$ and a conservative force becomes

$$-U'_{\rm eff} = -\frac{1}{2\lambda}\left(F(x)^2 - \int_0^1 F(x')^2\,{\rm d}x'\right) + o\left(\frac{1}{\lambda}\right). \tag{91}$$

The associated periodic potential $U_{\rm eff}$ reads

$$U_{\rm eff}(x) = \frac{1}{2\lambda}\left(\int_0^x F(x')^2\,{\rm d}x' - x\int_0^1 F(x')^2\,{\rm d}x'\right) + o\left(\frac{1}{\lambda}\right). \tag{92}$$

On the other side, it is instructive to determine how the effective force $F^{\rm eff}(x)$ is modified as the dynamical transition is approached: we illustrate in figure 3 how a cusp singularity appears in $F^{\rm eff}(x)$ as $\delta\lambda \to 0^+$ for $\lambda = \lambda_{\rm c} + \delta\lambda$ in the example system studied at the end of section 3.3. One sees from equation (86) that $F^{\rm eff}(x)$ is a regular function of x as long as $\delta\lambda > 0$ but develops a cusp singularity at its stationary points as $\delta\lambda \to 0^+$, explaining why the transition is not of the same nature as that of the standard depinning transition [56] (see [29] for a numerical study of how singularities in the effective force $F^{\rm eff}(x)$ are related to the dynamical phase transition, in the current large deviations of a particle a periodic sine potential).

To understand on a more general ground the relation between such depinning transition and the dynamical phase transition, we now consider the more generic case of an observable A with arbitrary g, $h \equiv 0$ and a force $F(x)$ presenting a stationary point x_0. We assume that F can be expanded around x_0 as $F(x) = (x - x_0)F_0 + o(x - x_0)$. In the effective dynamics, optimal trajectories are subjected to the effective force defined in equation (84) which reads as follows close to the stationary point[11]:

$$F^{\rm eff}(x) \simeq \sqrt{[(x - x_0)F_0]^2 + 2\phi(\lambda)}. \tag{93}$$

As the dynamical phase transition is approached ($\phi(\lambda) \to 0$ as $\lambda \to \lambda_{\rm c}^+$), this implies that the trajectories of the effective dynamics spend a longer and longer time close to x_0, meaning that the dynamics is mainly governed by the approximate form (93) of the effective force.

[11] We assume here that $\sigma = +1$ without loss of generality.

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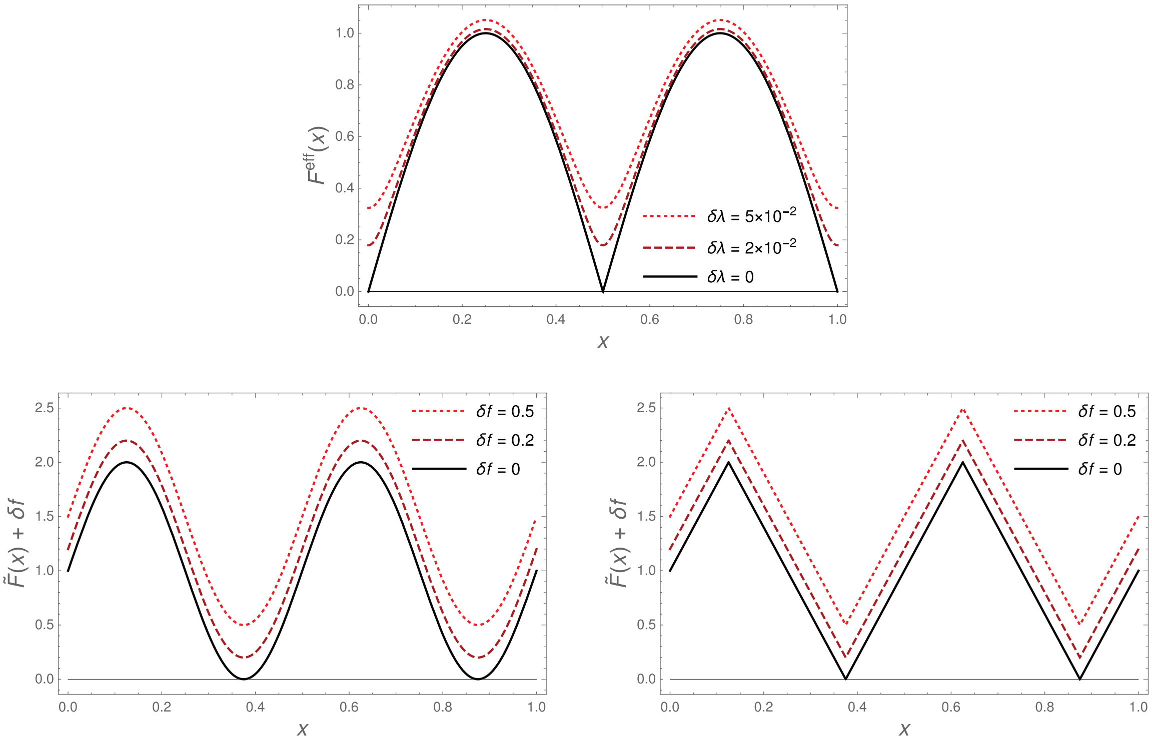

Figure 3. Comparison, on an example, of the criticality of the dynamical phase transition and of standard depinning transitions in 0d. (Top) The effective force $F^{\rm eff}(x)$ at $\lambda = \lambda_c + \delta\lambda$ deforms and becomes a cuspy function of x close to the stationary points that develop as $\delta\lambda \to 0^+$, for the example model studied at the end of section 3.3. (Bottom left) In the standard depinning transition in 0d, the depinning transition occurs when a regular force $\tilde{F}(x)$ possesses no stationary point any more when driven by a uniform force $\delta f > 0$. In this case, the force $\tilde{F}(x) + \delta f$ is a regular function of x and this implies that the transition is second-order [56], in contrast to our dynamical phase transition of interest. (Bottom right) If instead the force presents a cusp at all values of $\delta f \geqslant 0$ close to the $\delta f = 0$ stationary point, the transition becomes first-order [56], which is also different from our dynamical phase transition.

Effective trajectories are governed by the equation $\dot{x}(t) = F^{\rm eff}(x(t))$, whose solution reads

$$x(t) = x_0 \pm \sqrt{2\phi(\lambda)}\,\frac{\sinh(F_0 t)}{F_0}, \tag{94}$$

(up to an arbitrary translation in time) in the regime where the approximation (93) holds. As usual for the depinning transition in 0d problems [56], the average velocity of the trajectory (counted positively) is $|\bar{v}| \sim L/T$, with L the spatial period ($L = 1$ in our settings) and T the period in time. Close to the transition, the time period T is for instance estimated from

$$x(T/2) - x(-T/2) = L = 1 \tag{95}$$

(since $x(0) = x_0$ with our choice of the origin of time, so that $x(t) - x_0$ is an odd function of time). In the large-time limit, one finds from (95) that at dominant order in

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$\phi(\lambda) \to 0$, one has $T \sim \frac{1}{F_0}|\ln \phi(\lambda)|$. Consequently, the average velocity of the trajectory behaves as:

$$|\bar{v}| \sim \frac{F_0}{|\ln \phi(\lambda)|}, \tag{96}$$

in good agreement with the result (75) for the example of the $\sin(2\pi x)$ force. In all, we have shown that the behaviour (93) of the effective force close to the stationary point of the effective depinning problem governs the logarithmic form of the velocity close to the transition for an arbitrary current-type additive observable A.

Furthermore, if the additive observable A is the velocity of the particle, the relation $v(\lambda) = -\phi'(\lambda)$ together with (96) leads to:

$$\phi'(\lambda) \sim \frac{1}{|\ln \phi(\lambda)|} \qquad \text{for } \lambda \to \lambda_c^+ . \tag{97}$$

Close to the transition point, writing $\lambda = \lambda_c + \delta\lambda$, one can integrate this equation and get:

$$\phi(\lambda) \ln \delta\lambda - \phi(\lambda) = -\delta\lambda \tag{98}$$

so that at dominant order for $\delta\lambda \to 0^+$

$$\phi(\lambda) \sim \frac{\delta\lambda}{|\ln \delta\lambda|} \tag{99}$$

which is compatible with the result (74) obtained for the example model $F(x) = \sin(2\pi x)$.

In conclusion, the effective depinning transition problem fully describes the unexpected $\delta\lambda/|\ln \delta\lambda|$ behaviour (see equation (74)) of the expansion of $\phi(\lambda)$ near the transition, that we computed exactly for the sine potential, but now for the generic case when the effective force takes the form (93) close to its stationary point.

4.3. Interpretation of the effective process in the path-integral formulation

It is now interesting to come back to the path integral formulation introduced in section 2 to discuss the relationship between the effective non-equilibrium process and the original process biased by λ. To simplify the discussion, we specialise here to a conservative force $F(x) = -U'(x)$, so that we compare the effective non-equilibrium process with a λ-biased equilibrium process[12]. The aim of this subsection is to determine the possible relation between the action of the effective process and the biased action $S_\lambda[x(t), t_f]$, in order to see how the trajectories corresponding to these actions compare, in the weak-noise limit where the results of section 4.2 (that we will use) are valid.

A Langevin process with the effective non-equilibrium force $F^{\text{eff}}(x)$,

$$\dot{x}(t) = F^{\text{eff}}(x(t)) + \sqrt{\varepsilon}\,\eta(t)\,, \tag{100}$$

leads to an Onsager–Machlup action which, in the small-noise limit, takes the form

$$S_{\text{eff}}[x(t), t_f] = \int_0^{t_f} \frac{1}{2}\left(\dot{x} - F^{\text{eff}}(x)\right)^2 \mathrm{d}t\,. \tag{101}$$

[12] The generalisation to a non-conservative force field $F(x)$ is straightforward and left to the reader.

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Expanding the square in the action and using the expression (84) of the effective force $F^{\mathrm{eff}}(x)$, we end up with

$$S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]=\int_0^{t_{\mathrm{f}}}\left\{\frac{1}{2}\dot{x}^2+\frac{1}{2}F(x)^2+\lambda h(x)+\phi(\lambda)-\dot{x}F^{\mathrm{eff}}(x)\right\}\mathrm{d}t. \qquad (102)$$

Then, equations (84) and (85) imply that

$$\int_0^{t_{\mathrm{f}}}\dot{x}F^{\mathrm{eff}}(x)\,\mathrm{d}t=-\int_0^{t_{\mathrm{f}}}\lambda\dot{x}g(x)\,\mathrm{d}t \qquad (103)$$

so that the action of the non-equilibrium process reads

$$S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]=\int_0^{t_{\mathrm{f}}}\left\{\frac{1}{2}\dot{x}^2+\frac{1}{2}F(x)^2+\lambda\big(h(x)+\dot{x}g(x)\big)+\phi(\lambda)\right\}\mathrm{d}t. \qquad (104)$$

Finally, from the expression (9) of the biased action $S_\lambda[x(t),t_{\mathrm{f}}]$, one obtains

$$S_\lambda[x(t),t_{\mathrm{f}}]=S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]-\phi(\lambda)t_{\mathrm{f}}. \qquad (105)$$

This means that the action $S_\lambda[x(t),t_{\mathrm{f}}]$ of the biased process identifies, up to a constant, with the action $S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]$ of the effective non-equilibrium process. The difference $-\phi(\lambda)t_{\mathrm{f}}$ between these two actions has to be present since $S_\lambda[x(t),t_{\mathrm{f}}]$ describes a dynamics that does not preserve probability, while $S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]$ describes a probability-preserving one. The remarkable feature of (105) is that this difference is independent of the trajectory, so that a simple normalisation $S_\lambda[x(t),t_{\mathrm{f}}]+\phi(\lambda)t_{\mathrm{f}}$ of the biased action allows one to interpret it as the action of a probability-conserving process. This confirms that, in the weak-noise asymptotics we are working in, the effective non-equilibrium process defined by the force field $F^{\mathrm{eff}}(x)$ describes the same statistics of trajectories as the original dynamics biased by λ (after an adequate normalisation). Let us emphasise that in equation (105), the actions $S_{\mathrm{eff}}[x(t),t_{\mathrm{f}}]$ and $S_\lambda[x(t),t_{\mathrm{f}}]$ compare the non-equilibrium dynamics characterised by a force $F^{\mathrm{eff}}(x)=-U'_{\mathrm{eff}}(x)-\lambda$, with a λ-biased dynamics in the original conservative force field $F(x)=-U'(x)$.

4.4. Equivalence between the effective driven process and the λ-biased process

We have shown above, using the path-integral formalism in the specific case of a conservative force field $F(x)$, that the effective driven process described by $\mathbb{W}_\lambda^{\mathrm{eff}}$ is equivalent to the original λ-biased process described by $\mathbb{W}_\lambda$. We provide here for completeness a more general and formal proof of this equivalence in an operatorial formalism, following [20, 58, 73]. The force field $F(x)$ is here no longer assumed to derive from a potential. We also refer the reader to [21, 22] for a more mathematical description.

We start by defining a 'λ-ensemble' as a normalised average $\langle\cdot\rangle_\lambda^{[0,t_{\mathrm{f}}]}$ in the biased dynamics, namely

$$\langle\mathcal{O}[x(t)]\rangle_\lambda^{[0,t_{\mathrm{f}}]}=\frac{\left\langle\mathcal{O}[x(t)]\,\mathrm{e}^{-\frac{\lambda}{\varepsilon}A(t_{\mathrm{f}})}\right\rangle}{\left\langle\mathrm{e}^{-\frac{\lambda}{\varepsilon}A(t_{\mathrm{f}})}\right\rangle}, \qquad (106)$$

where the observable $\mathcal{O}$ depends on the trajectory. We made explicit the time interval on the lhs because the statistical properties of the λ-ensemble at times close to t_{f} are different from those in the 'bulk' of the time interval $[0, t_\mathrm{f}]$. The numerator of (106) corresponds to the biased process described by the operator $\mathbb{W}_\lambda$ while the denominator represents the proper normalisation that ensures $\langle 1 \rangle_\lambda^{[0,t_\mathrm{f}]} = 1$.

Let us now first focus on an observable $\mathcal{O}[x(t)] = \mathcal{O}_1(x(t_1))$ which depends only on the position of the particle at a time $t_1 \in [0, t_\mathrm{f}]$. Denoting by $\hat{\mathcal{O}}_1$ the diagonal operator whose components are the values of $\mathcal{O}_1(x)$, one has by definition

$$\left\langle \mathcal{O}_1(x(t_1)) \right\rangle_\lambda^{[0,t_\mathrm{f}]} = \frac{\langle - | \mathrm{e}^{(t_\mathrm{f}-t_1)\mathbb{W}_\lambda} \, \hat{\mathcal{O}}_1 \, \mathrm{e}^{t_1 \mathbb{W}_\lambda} | P_\mathrm{i} \rangle}{\langle - | \mathrm{e}^{t_\mathrm{f} \mathbb{W}_\lambda} | P_\mathrm{i} \rangle} \, . \tag{107}$$

Then, if both t_1 and $t_\mathrm{f} - t_1$ are large compared to the inverse of the gap of $\mathbb{W}_\lambda$, that is to say, if t_1 is in the bulk of the time interval $[0, t_\mathrm{f}]$, one can use the asymptotic behaviour (14) of $\mathrm{e}^{t\mathbb{W}_\lambda}$, leading to

$$\left\langle \mathcal{O}_1(x(t_1)) \right\rangle_\lambda^{[0,t_\mathrm{f}]} \underset{t_\mathrm{f}\to\infty}{\longrightarrow} \langle L | \hat{\mathcal{O}}_1 | R \rangle = \int \mathcal{O}_1(x) L(x) R(x) \, \mathrm{d}x \, . \tag{108}$$

In other words, as well known [22, 58, 73], the intermediate-times λ-ensemble statistics is governed by the product of the left- and right-eigenvectors of $\mathbb{W}_\lambda$.

Consider now the effective dynamics described by the operator $\mathbb{W}_\lambda^\mathrm{eff}$. From its definition (76), one sees that the left- and right-eigenvectors associated with its largest eigenvalue 0 are respectively equal to $\langle - |$ and $| LR \rangle$. In analogy with (14), the large-time behaviour of the propagator $\mathrm{e}^{t_\mathrm{f} \, \mathbb{W}_\lambda^\mathrm{eff}}$ is thus given by

$$\mathrm{e}^{t_\mathrm{f} \, \mathbb{W}_\lambda^\mathrm{eff}} \underset{t_\mathrm{f}\to\infty}{\sim} | LR \rangle \langle - |, \tag{109}$$

where $| LR \rangle \equiv \hat{L} | R \rangle$. Hence, similarly to (108), one finds that the average $\langle \cdot \rangle_\mathrm{eff}^{[0,t_\mathrm{f}]}$ of an observable for the effective dynamics is given in the steady state by

$$\left\langle \mathcal{O}_1(x(t_1)) \right\rangle_\mathrm{eff}^{[0,t_\mathrm{f}]} \underset{t_\mathrm{f}\to\infty}{\longrightarrow} \langle - | \hat{\mathcal{O}}_1 | LR \rangle = \int \mathcal{O}_1(x) L(x) R(x) \, \mathrm{d}x, \tag{110}$$

which is equal to the corresponding λ-ensemble average (108). The statistical properties of the biased dynamics, described by (106), are thus equal to those of the effective dynamics at any time t_1 in the bulk of $[0, t_\mathrm{f}]$.

To go further and understand the equivalence at a trajectorial level, one has to consider more than one-time observables. Jack and Sollich [20] considered for instance a time-discrete settings and worked at the level of trajectory probabilities, showing that in the bulk of $[0, t_\mathrm{f}]$ there is an equivalence between the trajectory probabilities of the effective process and of the (normalised) biased dynamics. Equivalently, one can formulate this equivalence using multi-time correlation functions of arbitrary observables (but now in continuous time) as follows. Using the identity $\mathrm{e}^{t \, \mathbb{W}_\lambda^\mathrm{eff}} = \mathrm{e}^{-t \, \varphi_\varepsilon(\lambda)} \, \hat{L} \, \mathrm{e}^{t \, \mathbb{W}_\lambda} \hat{L}^{-1}$ inferred from the definition (76) of the effective operator, the previous reasoning can be readily extended to observables of the form $\mathcal{O}[x(t)] = \mathcal{O}_1(x(t_1)) \mathcal{O}_2(x(t_2)) \ldots \mathcal{O}_n(x(t_n))$ depending on the position of the particle at different times t_1, t_2, …, t_n which are all in the bulk of the interval $[0, t_\mathrm{f}]$ (but which can be arbitrarily close to each other). One finds:

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$$\left\langle O_1(x(t_1))\mathcal{O}_2(x(t_2))\ldots\mathcal{O}_n(x(t_n))\right\rangle_{\text{eff}}^{[0,t_\text{f}]} \underset{t_1,t_\text{f}\to\infty}{\sim} \left\langle O_1(x(t_1))\mathcal{O}_2(x(t_2))\ldots\mathcal{O}_n(x(t_n))\right\rangle_{\lambda}^{[0,t_\text{f}]} \tag{111}$$

$$= \langle -|\mathcal{O}_n \mathrm{e}^{(t_n-t_{n-1})\mathbb{W}_\lambda^{\text{eff}}}\mathcal{O}_{n-1}\ldots\mathrm{e}^{(t_2-t_1)\mathbb{W}_\lambda^{\text{eff}}}\hat{\mathcal{O}}_1|LR\rangle. \tag{112}$$

This corresponds to the notion of trajectorial asymptotic equivalence, developed by Chetrite and Touchette [21, 22], between the biased ensemble (106) and the effective dynamics.

5. Conclusion and outlook

In this work we have identified an effective probability-conserving dynamics turning the rare trajectories of a stochastic process into the typical histories of an explicit modified dynamics, in the case of a particle diffusing in a periodic one-dimensional generic force under a weak thermal noise. In this way, by using large-deviation techniques, we have determined the form of the force that a particle effectively withstands when conditioned to bear an atypical fluctuation for a long duration. Interestingly, the resulting effective non-equilibrium process does not only differ from the original λ-biased dynamics by the addition of a (λ-dependent) uniform driving force, but the conservative part of the force is also 'renormalised' by the presence of λ, even in the simple case when the observable $A(t_\text{f})$ is the average current. Considering this result from a reversed perspective, we also learn that a particle in a potential driven by a uniform non-conservative force cannot be accurately represented by a λ-biased dynamics in the same potentials. This means that a tentative statistical approach that would try to evaluate (by analogy with equilibrium) mean values of physical observables by taking a flat average over configurations with the same current (as defined by the λ-biased dynamics) would be at best an approximation.

Along the way, we have analysed the fluctuations of time-integrated current-type observables in a periodic system. These display a rich phenomenology associated with the existence of dynamical phase transitions between a static fluctuating phase, characterised by a flat SCGF, and a phase with time-periodic travelling trajectories, associated to a SCGF being equal to the energy of a natural optimisation problem—which takes the form of a conservative Hamiltonian dynamics. Determining analytically how a finite noise rounds the observed transition (see for instance [29]) is also an interesting open question. Furthermore, we have described an alternative way to compute the SCGF without using the variational techniques derived from the weak-noise analysis of the path-integral representation, that allowed us to show how the large-time and the weak-noise limits commute.

The obtained results also open a direction of research to characterise fluctuations in a given system by engineering a new system subjected to an additional external force. Such an approach has been used in recent studies on adaptive algorithms, based for instance on a feedback procedure to evaluate the effective force [73–76], improving the computational efficiency. The weak-noise regime has been seldom studied (it is in fact known to present specific difficulties [73]), and the results we present in this paper could help to understand large-time fluctuations and their associated phenomenology both in experiments and simulations.

Remark. In the final stage of the preparation of this manuscript, we became aware that works parallel to ours were completed using similar approaches [77, 78]. The results of [77] are complementary to ours and elucidate interesting finite-time behaviours in the initial and final times of the observation window $[0, t_f]$, that we have not studied. In [78], the authors determine the non-zero temperature behaviour of the SCGF, for a periodic potential that presents no local minima.

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