@Nice

Inactive Dynamical Phase of an Exclusion Process on a Ring

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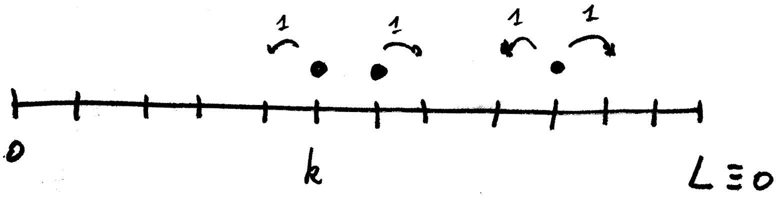
Refs: VL, JP Garrahan, F van Wijland JPA 2012
C Appert-Rolland, B Derrida, VL, FvW PRE 2008

Aim:
- Study (large) deviations of a dynamical observable
  Characterize the phase-transition in its distribution, through observables.
- Provide correspondences btw tools for stoch. processes
- Translate results into completely different languages

SYSTEM: Simple Symmetric Exclusion Process (SSEP)



jumps with rate 1 to neighboring site, provided: target site is empty (exclusion rule)

periodic boundary conditions

L sites
N particles

$\rho_0 = \frac{N}{L}$ = density

L & N fixed
t large

Focus on the "dynamical activity" K = # events during a given history

$$K = \#\,\overset{\curvearrowright}{\text{jumps}} + \#\,\overset{\curvearrowleft}{\text{jumps}}$$

(cf $Q = \#\,\overset{\curvearrowright}{\text{jumps}} - \#\,\overset{\curvearrowleft}{\text{jumps}}$ for the current) [Kirone's notation: Y_t]

$$P(K,t) \overset{t\to\infty}{\sim} e^{\pm t\, \pi_L(K/t)}$$ [Depends on more than the mere steady state]

$$\underbrace{\langle e^{-sK}\rangle}_{\text{average over histories of duration } t} \sim e^{+t\,\psi_L(s)}$$

$t\,\psi(s)$ = cumulant generating function of K

TOOL 1: OPERATOR of EVOLUTION & LINK TO QUANTUM SYSTEMS

- Master equation:

$$\partial_t P(\overbrace{n_1 \dots n_L}^{\vec{n}}, t) = \sum_k (\quad) P(\dots n_k - 1 \; n_k + 1 \dots \qquad \text{on each site}$$

Vector formulation: $|P(\vec{n},t)\rangle = \sum_{\vec{n}} P(\vec{n},t) \, |\vec{n}\rangle$

$$\begin{cases} S^+ |n\rangle = (1-n) |n+1\rangle \\ S^- |n\rangle = n |n-1\rangle \\ \hat{n} \, |n\rangle = n |n\rangle \end{cases}$$

$$\partial_t |P(\vec{n},t)\rangle = \mathbb{W} \, |P(\vec{n},t)\rangle$$

$$\mathbb{W} = \sum_{k=1}^{L} S_k^{+} S_{k+1}^{+} + S_k^{-} S_{k+1}^{+} + \hat{n}_k (1-\hat{n}_{k+1}) + \hat{n}_{k+1} (1-\hat{n}_k)$$

$$= \underbrace{\sum_k S_k^x S_{k+1}^x + {}^{y\,y} + {}^{z\,z}}_{-\text{Quantum Hamiltonian XXX spin chain}} + \underline{\text{constant}}_{N,L}$$

- Including s: $\hat{P}(s,\vec{n},t) = \sum_K e^{-sK} P(n,K,t)$ $\quad$ eg: $\langle e^{-sK} \rangle = \sum_{\vec{n}} \hat{P}(s,\vec{n},t)$

$$\partial_t |\hat{P}\rangle = \mathbb{W}(s) \, |\hat{P}\rangle \qquad \to \psi(s) = \max \text{Spectrum } \mathbb{W}(s)$$

$$\mathbb{W}(s) = \sum_k e^{-s} (S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+) + \hat{n}_k (1-\hat{n}_{k+1}) + (k \leftrightarrow k+1)$$

$$= e^{-s} \underbrace{\sum_k \left(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + \Delta S_k^z S_{k+1}^z \right)}_{\text{anisotropic XXZ spin chain}} + \text{constant}_{N,L} \;, \quad \Delta = e^{s}$$

→ large deviation of SSEP ⟷ energy level of a quantum chain at fixed magnetization

- Expected result:

$s \to -\infty$ active histories $\qquad$ ○● ○● ○● ○● (Luttinger liquid)

$s \to 0$ standard histories $\qquad$ flat profile

$s \to +\infty$ inactive histories $\qquad$ ○○○○●●●●●

phase separated Ferro mag.

TOOL 2. LARGE SYSTEM SIZE LIMIT & LINK TO MACROSCOPIC FLUCTUATION THEORY (MFT)

SU(2) coherent states

$$\langle e^{-sK}\rangle = \langle - | e^{t\mathbb{W}(s)} | \cdots \rangle \overset{\downarrow}{=} \int \mathcal{D}\rho\, \mathcal{D}\hat\rho \; e^{-LS[\rho,\hat\rho;s]}$$

In principle: valid for finite L.

Macroscopic scalings: $(L\to\infty)$

$x = k/L \in [0,1]$

$\tau = t/L^2$

$\lambda = s/L^{-2}$

+ Gradient Expansion

$\sigma(\rho) = 2\rho(1-\rho)$

$\rho(x,t)$

$$S[\rho,\hat\rho;\lambda] = \int_0^1 dx \int_0^T d\tau \left\{ \hat\rho\left(\partial_\tau\rho - \Delta\rho\right) - \frac{\sigma(\rho)}{2}(\partial_x\hat\rho)^2 + \lambda\sigma(\rho) \right\}$$

↑ quadratic in $\hat\rho$

$$\underbrace{+\mu\int_0^1 dx\,(\rho-\rho_0)}_{\text{to fix the mean density}}$$

↳ [Otherwise, the optimization process of saddle-point equations also selects a density $\neq \rho_0$]

Remark: integrating over $\partial\hat\rho$ and introducing j solution of $\partial_t\rho + \nabla j = 0$ one recovers the roman MFT (cf Krzysztof's & Kirone's talk)

large L limit: saddle-point evaluation ; optimal trajectory

SIMPLE CASE: uniform steady optimal profile trajectory

$$\psi(s) = -s\frac{\langle K\rangle}{t} = -\frac{1}{L}s\,\sigma(\rho_0)$$

] in direct space ($P(K,t)$) the result is trivial ↳ one needs more terms & finite size study –

no other term than the mean!

FINITE SIZE STUDY:

$$\mathcal{F}(x) = 2x - 4\sqrt{2}\,x^{3/2}\int_{-1}^{1} dy\, y^2 \cos\left[\sqrt{2x}\sqrt{1-y^2}\right]$$

$$\varphi(\lambda) = -\frac{1}{L}s\,\sigma(\rho_0) + \frac{1}{L^2}\underbrace{\mathcal{F}\left(-\frac{\sigma(\rho_0)\,\sigma''(\rho_0)}{8}\lambda\right)}$$

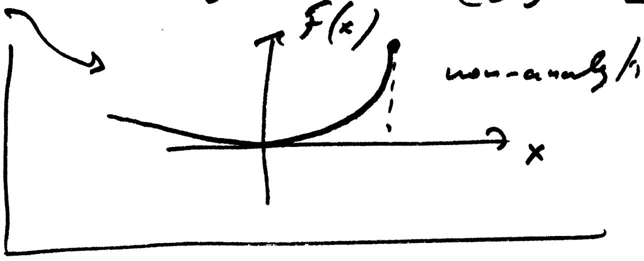
arises from either

- integration of Gaussian fluctuation around saddle
- Bethe Ansatz: $\frac{1}{L}$ <=> Bethe roots are along a path in $\mathbb{C}$; $\frac{1}{L^2}$ <=> Effects of the discreteness of the roots

⚠ SAME RESULT ⚠

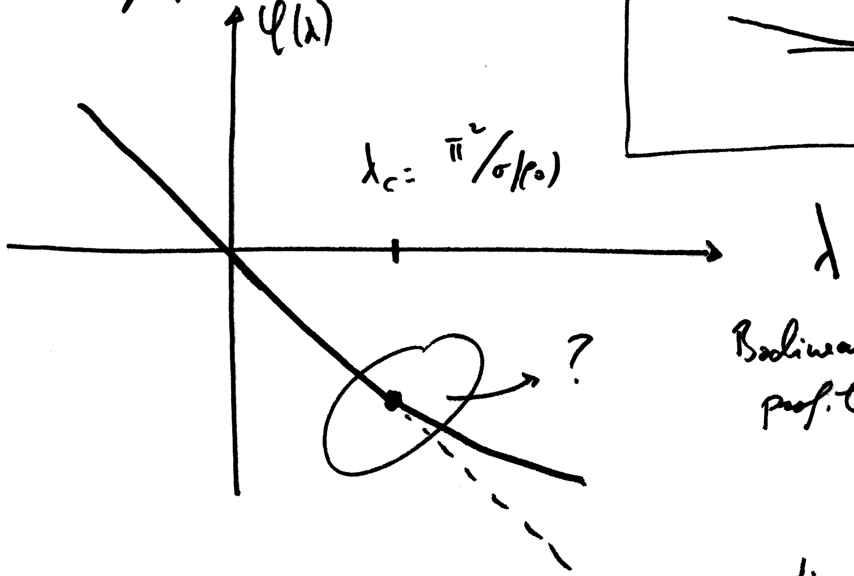
PHASE TRANSITION:

$F(x) = C_1 - C_2\sqrt{\pi^2/2 - x}$ $x \to \left(\frac{\pi^2}{2}\right)^-$ @Nice



non-analytic

The uniform profile is unstable



covered in WASEP

Bodineau & Derrida : around λ_c

profile

Saddle point equation, now for $\rho(x)$ [time-independent
space-dependent

$$\mathcal{L}[\rho] = \frac{(\partial_x\rho)^2}{2\sigma} + \lambda\sigma + (\rho-\rho_0)\mu \qquad \frac{\partial\mathcal{L}}{\partial\rho} = 0 \quad \partial\frac{\partial\mathcal{L}}{\partial(\partial\rho)} = 0$$

$\mathcal{H}[\rho] = (-) \; -(-) - (-)$ is conserved

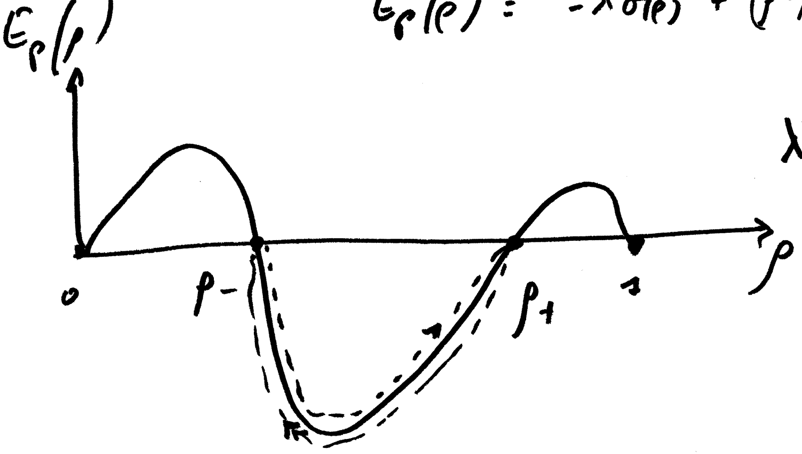
Equation of "motion" for $\rho(x)$

"position" variable

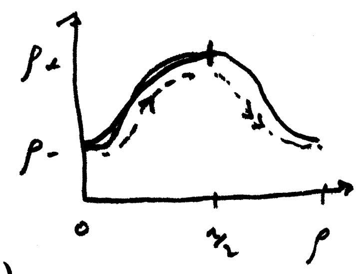
"time" variable

Formulation: $\frac{1}{2}(\partial_x\rho)^2 + \underbrace{E_p(\rho)} = cst \equiv E_0$

$E_p(\rho) = -\lambda\sigma(\rho)^2 + (\rho-\rho_0)\cdot\mu\cdot\sigma(\rho)$ quartic in ρ



$\lambda > \lambda_c$



Conditions: $\rho_0 = \int_0^1 dx\, \rho(x)$ $\rho(0) = \rho(1)$

$$1 = \int_0^1 dx = 2\int_{\rho_-}^{\rho_+} d\rho\, \frac{dx}{d\rho}$$

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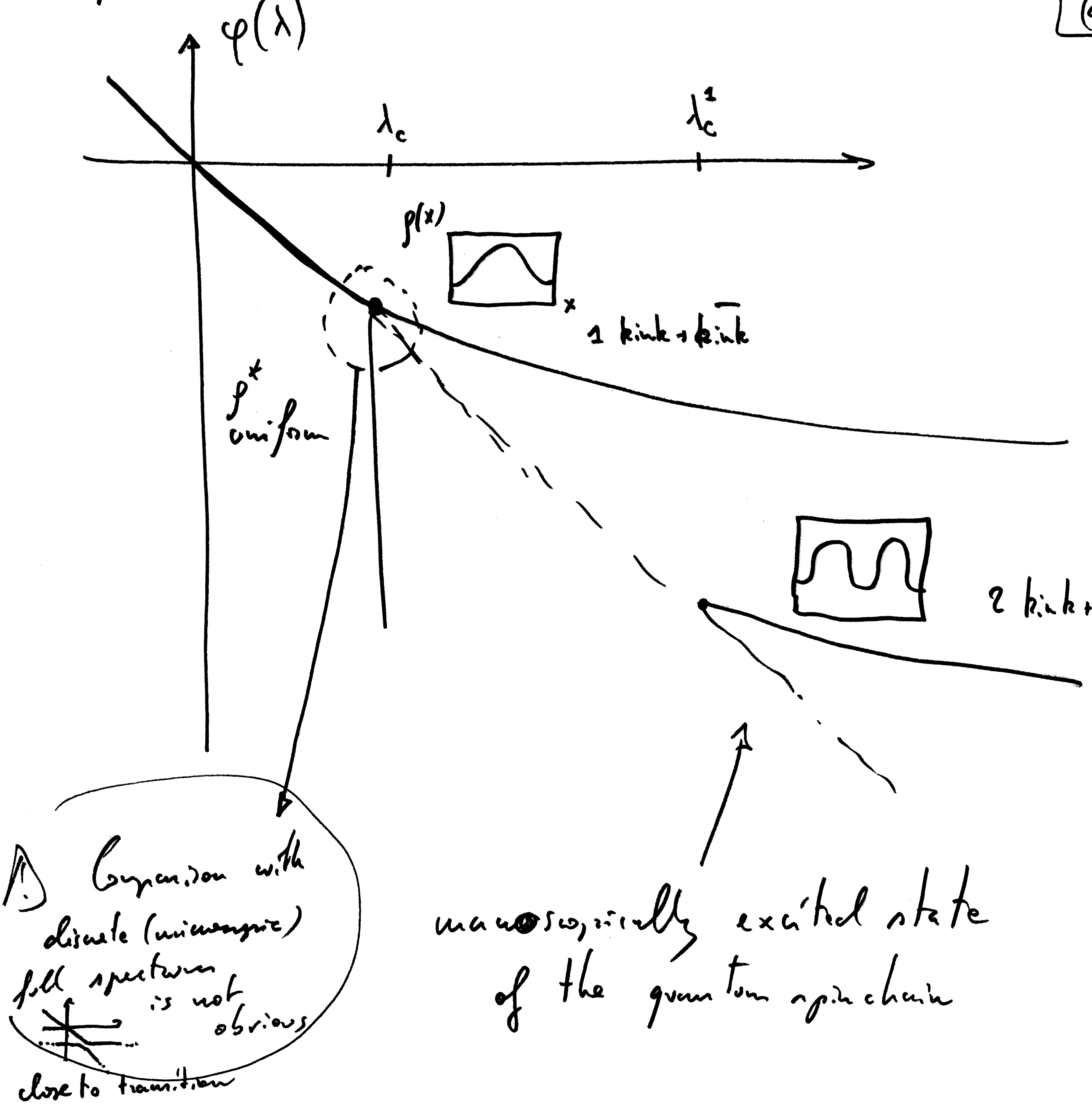
Full picture:



⚠ Comparison with discrete (microscopic) full spectrum is not obvious close to transition

macroscopically excited state of the quantum spin chain

Current work with M. Cheneau & T. Giamarchi

Open questions:

- how does this translate | to Bethe Ansatz (repartition of roots) / to quantum systems (isolated)
- WASEP with large deviation of the current ← [illegible] transition
- ASEP with KPZ (ie non-diffusive) scalings
- quantum systems with finite temperature (⇔ finite time) SSEP