# Symmetries in large deviations of additive observables: what one gains from forgetting probabilities and turning to the quantum world


Marc Cheneau[1], Juan P. Garrahan[2], Frédéric van Wijland[3]

Cécile Appert-Rolland[4], Bernard Derrida[5], Alberto Imparato[6]

[1]Institut d'Optique, Palaiseau [2]Nottingham University [3]MSC, Paris
[4]LPT, Orsay [5]LPS, ENS, Paris [6]Aarhus University


Nice – June 9th 2015

# Classical and quantum dynamics

- Correspondence
  - generator of **stochastic** classical system [particles hopping]
  - Hamiltonian of **quantum** XXZ chain

  (Well known at least in the stat. mech. community.)

- Use: dictionary between
  - regimes of **large deviations** of *dynamical* observables
  - phases across a Quantum Phase Transition

# Classical and quantum dynamics

- Correspondence
  - generator of **stochastic** classical system [particles hopping]
  - Hamiltonian of **quantum** XXZ chain

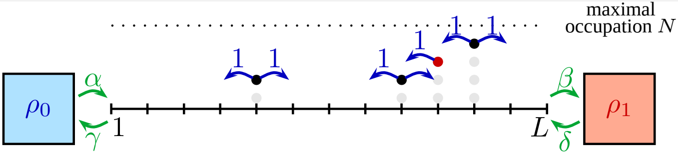
  (Well known at least in the stat. mech. community.)

- Use: dictionary between
  - regimes of **large deviations** of *dynamical* observables
  - phases across a Quantum Phase Transition

- Perspectives opened ; questions raised
  - finite-size effects
  - large/small scale spectrum
  - **import/export techniques from/to stat. mech.**

  (I will ask questions to *you*.)

# Exclusion Processes – generic settings




- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution for the **probability** $P(\{n_i\}, t)$

$$\partial_t P(\{n_i\}, t) = \sum_{n_i'} \left[ W(n_i' \to n_i) P(\{n_i'\}, t) - W(n_i \to n_i') P(\{n_i\}, t) \right]$$

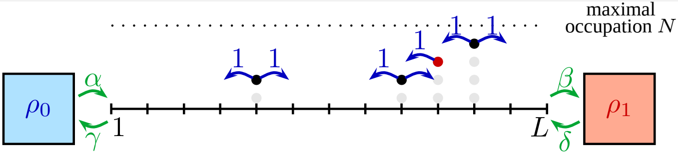
- **Large deviation function** of "additive" observables $A$

$$\langle e^{-sA} \rangle \sim e^{t\,\psi(s)} \qquad (\Leftrightarrow \text{determining } P(A,t))$$

$A$ = total current $Q$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} - \overleftarrow{\text{jumps}}$

$A$ = total activity $K$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} + \overleftarrow{\text{jumps}}$

# Exclusion Processes – generic settings




- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution for the **probability** $P(\{n_i\}, t)$

$$\partial_t P(\{n_i\}, t) = \sum_{n_i'} \left[ W(n_i' \to n_i) P(\{n_i'\}, t) - W(n_i \to n_i') P(\{n_i\}, t) \right]$$

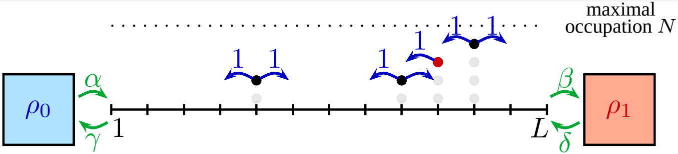
- **Large deviation function** of "additive" observables $A$

$$\langle e^{-sA} \rangle \sim e^{t\,\psi(s)} \qquad (\Leftrightarrow \text{determining } P(A, t))$$

$A =$ total current $Q$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} - \overleftarrow{\text{jumps}}$

$A =$ total activity $K$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} + \overleftarrow{\text{jumps}}$

# Exclusion Processes – generic settings




- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \le n_i \le N$
- Markov evolution for the **probability** $P(\{n_i\}, t)$

$$\partial_t P(\{n_i\}, t) = \sum_{n_i'} \left[ W(n_i' \to n_i) P(\{n_i'\}, t) - W(n_i \to n_i') P(\{n_i\}, t) \right]$$

- **Large deviation function** of "additive" observables $A$

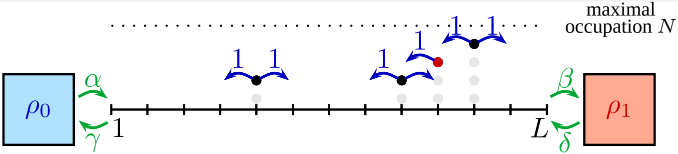
$$\langle e^{-sA} \rangle \sim e^{t\,\psi(s)} \qquad (\Leftrightarrow \text{determining } P(A,t))$$

$A =$ total current $Q$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} - \overleftarrow{\text{jumps}}$

$A =$ total activity $K$ on time window $[0, t]$ $\quad = \#\overrightarrow{\text{jumps}} + \overleftarrow{\text{jumps}}$

# Operator representation

[Schütz & Sandow PRE **49** 2726]




Evolution of probability vector $P$:

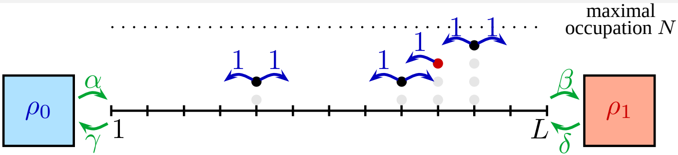
$$\partial_t P = \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\leq k\leq L-1} \left[\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \hat{n}_{k+1}\check{n}_k\right]$$
$$+ \alpha \left[\sigma_1^+ - \check{n}_1\right] + \gamma \left[\sigma_1^- - \hat{n}_1\right]$$
$$+ \delta \left[\sigma_L^+ - \check{n}_L\right] + \beta \left[\sigma_L^- - \hat{n}_L\right]$$

$\sigma^\pm = \sigma^x \pm i\sigma^y$ and $\sigma^z = \hat{n} - \frac{N}{2}$ are spin operators (of "spin" $j = \frac{N}{2}$)

# Operator representation

[Schütz & Sandow PRE **49** 2726]




Evolution of probability vector $P$:

$$\partial_t P = \mathbb{W}\, P$$

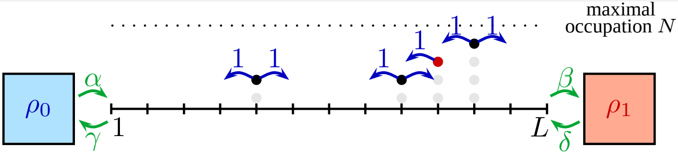
$$\mathbb{W} = \sum_{1\leq k\leq L-1} \left[\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ - \hat{n}_k\check{n}_{k+1} - \hat{n}_{k+1}\check{n}_k\right] + \alpha\left[\sigma_1^+ - \check{n}_1\right] + \gamma\left[\sigma_1^- - \hat{n}_1\right] + \delta\left[\sigma_L^+ - \check{n}_L\right] + \beta\left[\sigma_L^- - \hat{n}_L\right]$$

$\sigma^\pm = \sigma^x \pm i\sigma^y$ and $\sigma^z = \hat{n} - \frac{N}{2}$ are spin operators (of "spin" $j = \frac{N}{2}$)

densities $\rho_0 = \frac{\alpha}{\alpha+\gamma}$ ; $\rho_1 = \frac{\delta}{\delta+\beta}$ ; contact rates $a_0 = \frac{\alpha}{\gamma}$ ; $a_1 = \frac{\delta}{\beta}$

# Operator representation

[Schütz & Sandow PRE **49** 2726]




Evolution of probability vector $P$:

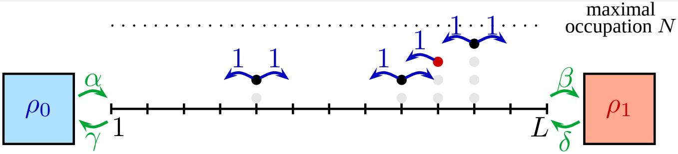
$$\partial_t P = \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\leq k\leq L-1} \left[\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \hat{n}_{k+1}\check{n}_k\right] + \alpha\left[\sigma_1^+ - \check{n}_1\right] + \gamma\left[\sigma_1^- - \hat{n}_1\right] + \delta\left[\sigma_L^+ - \check{n}_L\right] + \beta\left[\sigma_L^- - \hat{n}_L\right]$$

$\sigma^\pm = \sigma^x \pm i\sigma^y$ and $\sigma^z = \hat{n} - \frac{N}{2}$ are spin operators (of "spin" $j = \frac{N}{2}$)

**XXX spin chain Hamiltonian** (up to boundary terms and constants).

# Operator representation for **large deviations**

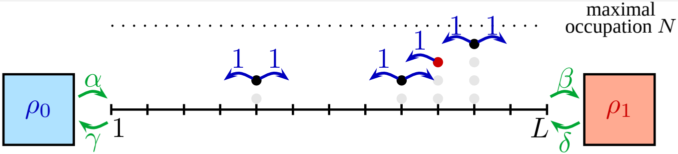



$$\left\langle \mathrm{e}^{-sK} \right\rangle \sim \mathrm{e}^{t\psi(s)} \qquad \text{with} \qquad \psi(s) = \max \operatorname{Sp} \mathbb{W}_s$$

$$\mathbb{W}_s = \sum_{1 \le k \le L-1} \left[ \mathrm{e}^{-s} \sigma_k^+ \sigma_{k+1}^- + \mathrm{e}^{-s} \sigma_k^- \sigma_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \hat{n}_{k+1} \check{n}_k \right]$$
$$+ \alpha \left[ \mathrm{e}^{-s} \sigma_1^+ - \check{n}_1 \right] + \gamma \left[ \mathrm{e}^{-s} \sigma_1^- - \hat{n}_1 \right]$$
$$+ \delta \left[ \mathrm{e}^{-s} \sigma_L^+ - \check{n}_L \right] + \beta \left[ \mathrm{e}^{-s} \sigma_L^- - \hat{n}_L \right]$$

for the **activity** $K$: **XXZ spin chain Hamiltonian**

# Operator representation for **large deviations**




$$\left\langle e^{-sQ} \right\rangle \sim e^{t\psi(s)} \qquad \text{with} \qquad \psi(s) = \max \operatorname{Sp} \mathbb{W}_s$$

$$\mathbb{W}_s = \sum_{1 \le k \le L-1} \left[ e^{+s} \sigma_k^+ \sigma_{k+1}^- + e^{-s} \sigma_k^- \sigma_{k+1}^+ - \hat{n}_k \check{n}_{k+1} - \hat{n}_{k+1} \check{n}_k \right]$$
$$+ \alpha \left[ e^{-s} \sigma_1^+ - \check{n}_1 \right] + \gamma \left[ e^{+s} \sigma_1^- - \hat{n}_1 \right]$$
$$+ \delta \left[ e^{+s} \sigma_L^+ - \check{n}_L \right] + \beta \left[ e^{-s} \sigma_L^- - \hat{n}_L \right]$$

for the **current** $Q$: **“asymmetric” XXZ spin chain Hamiltonian**

Example 1: use of rotational symmetry

map non-equilibrium current fluctuations

to equilibrium current fluctuations

# Mapping non-eq to eq

[Imparato, VL, van Wijland, PTPS 184 276 ]

## Large deviations of the current

$$\psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

$$\mathbb{W}(s) = \overbrace{\sum_{1\le k\le L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}} + \text{ constant}$$
$$+ \alpha \left[S_1^+ - \check{n}_1\right] + \gamma \left[S_1^- - \hat{n}_1\right]$$
$$+ \delta \left[S_L^+ \mathrm{e}^{s} - \check{n}_L\right] + \beta \left[S_L^- \mathrm{e}^{-s} - \hat{n}_L\right]$$

# Mapping non-eq to eq

[Imparato, VL, van Wijland, PTPS 184 276 ]

## Large deviations of the current

$\psi(s) = \max \mathrm{Sp}\, \mathbb{W}(s)$

$$\mathbb{W}(s) = \overbrace{\sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}} \qquad \text{[non-Hermitian due to boundaries]}$$
$$+ \alpha \left[S_1^+ - \check{n}_1\right] + \gamma \left[S_1^- - \hat{n}_1\right]$$
$$+ \delta \left[S_L^+ e^{s} - \check{n}_L\right] + \beta \left[S_L^- e^{-s} - \hat{n}_L\right]$$

## Local transformation

$$\mathcal{Q}^{-1}\mathbb{W}(s)\mathcal{Q} = \sum_{1\leq k\leq L-1} \vec{S}_k \cdot \vec{S}_{k+1}$$
$$+ \alpha' \left[S_1^+ - \check{n}_1\right] + \gamma' \left[S_1^- - \hat{n}_1\right]$$
$$\underbrace{+\delta' \left[S_L^+ e^{s'} - \check{n}_L\right] + \beta' \left[S_L^- e^{-s'} - \hat{n}_L\right]}_{\text{describes contact with reservoirs of same densities}}$$

## SO(3) symmetry [Imparato, VL, van Wijland, PTPS 184 276 ]

Detailed transformation: (on **one** site)

$$\mathcal{Q} = \mathbf{1} + xS^x - \mathrm{i}yS^y + zS^z \qquad \text{(invertible)}$$

performs a **rotation** of the vector $\mathbf{S} = (S^x, S^y, S^z)$ of spin operators

$$\mathcal{Q}^{-1}S^x\mathcal{Q} = (R\mathbf{S})_1 \qquad \mathcal{Q}^{-1}S^y\mathcal{Q} = (R\mathbf{S})_2 \qquad \mathcal{Q}^{-1}S^z\mathcal{Q} = (R\mathbf{S})_3$$

for some SO(3) rotation matrix $R$.

# SO(3) symmetry [Imparato, VL, van Wijland, PTPS 184 276 ]

Detailed transformation: (on **one** site)

$$\mathcal{Q} = \mathbf{1} + xS^x - \mathrm{i}yS^y + zS^z \qquad \text{(invertible)}$$

performs a **rotation** of the vector $\mathbf{S} = (S^x, S^y, S^z)$ of spin operators

$$\mathcal{Q}^{-1} S^x \mathcal{Q} = (R\mathbf{S})_1 \qquad \mathcal{Q}^{-1} S^y \mathcal{Q} = (R\mathbf{S})_2 \qquad \mathcal{Q}^{-1} S^z \mathcal{Q} = (R\mathbf{S})_3$$

for some SO(3) rotation matrix $R$.

Form of the matrix: (Cayley form)

$$R = (I + A)(I - A)^{-1}$$

$$A = \begin{pmatrix} 0 & -iz & y \\ iz & 0 & -ix \\ -y & ix & 0 \end{pmatrix}$$

# Large deviations

[Imparato, VL, van Wijland, PTPS 184 276 ]

Result: (transforming **all** sites)

$$\mathcal{Q}^{-1}\mathbb{W}_{\mathsf{res}}(s;\rho_0,\rho_1;a_0,a_1)\,\mathcal{Q} = \mathbb{W}_{\mathsf{res}}(s';\rho_0',\rho_1';a_0,a_1)$$

with “primed” variables

$$\rho_0' = \frac{(1+x)\rho_0 - x - z}{1-x}$$

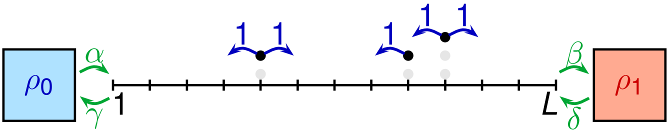
$$\rho_1' = (x+e^{-s} - z(1-e^{-s}))\,\frac{\big[x+e^{s}+z(1-e^{s})\big]\rho_1\; - x - z}{1-x^2}$$

$$e^{-s'} = \frac{x+e^{-s}+z(e^{-s}-1)}{1+xe^{-s}+z(e^{-s}-1)}$$

# Summary

[Imparato, VL, van Wijland, **PRE** 80 011131]

Symmetric exclusion process

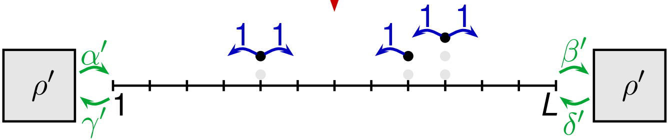



Local transformation

$\mathrm{Prob}^{\text{stationn.}}_{\text{non-eq}}(\text{current})$

$\updownarrow$

$\mathrm{Prob}^{\text{stationn.}}_{\text{eq}}(\text{current}')$




**System in equilibrium**

# Probabilistic interpretation

Measure $\hat{P}(\mathbf{n}, s, t)$ biased by $e^{-sQ}$

Mapping:

$$\begin{aligned} &\hat{P}(\mathbf{n}, s, t; \rho_0, \rho_1; a_0, a_1) \\ &= \langle \mathbf{n} | e^{t\mathbb{W}(s;\rho_0,\rho_1;a_0,a_1)} | P_{\text{init}} \rangle \\ &= \underbrace{\langle \mathbf{n} | \mathcal{Q}}_{\substack{\text{new projection} \\ \text{state}}} \, e^{t\mathbb{W}(s';\rho_0',\rho_1';a_0,a_1)} \underbrace{\mathcal{Q}^{-1} | P_{\text{init}} \rangle}_{\substack{\text{new initial} \\ \text{condition}}} \end{aligned}$$

# Probabilistic interpretation

Measure $\hat{P}(\mathbf{n}, s, t)$ biased by $e^{-sQ}$

**Mapping:**

$$\begin{aligned}\hat{P}(\mathbf{n},s,t;\rho_0,\rho_1;a_0,a_1) &= \langle\mathbf{n}|e^{t\mathbb{W}(s;\rho_0,\rho_1;a_0,a_1)}|P_{\text{init}}\rangle \\ &= \underbrace{\langle\mathbf{n}|\mathcal{Q}}_{\text{new projection state}}\, e^{t\mathbb{W}(s';\rho_0',\rho_1';a_0,a_1)}\, \underbrace{\mathcal{Q}^{-1}|P_{\text{init}}\rangle}_{\text{new initial condition}}\end{aligned}$$

**Question:**

What is the mathematical embedding (in terms of process&prob.)?
(Duality, Radon-Nykodym? **caveat**: prob. not preserved)

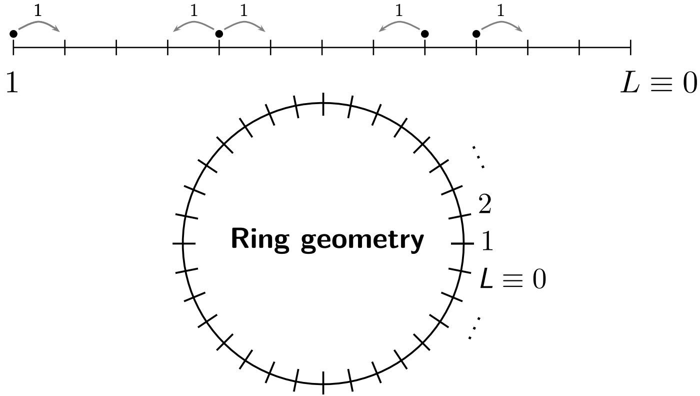
**Generalization:**

- ⋆ higher dimensions
- ⋆ generic network and current
- ⋆ more than two reservoirs
- ⋆ see also: Derrida & Gerschenfeld ($\omega$ variable)
  Akkermans, Bodineau, Derrida & Shpielberg (1d LDF for $d > 1$)

# Example $2$: exclusion process on a ring

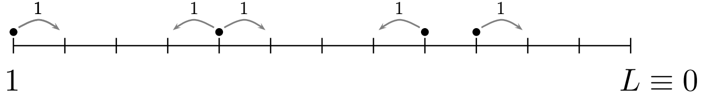
# Focus on a simple situation

**Simple** exclusion process (SSEP): maximal occupation $N = 1$
Periodic boundary conditions
Fixed total particle number $N_0$ density: $\rho_0 = N_0/L$



# Focus on a simple situation

**Simple** exclusion process (SSEP): maximal occupation $N = 1$
Periodic boundary conditions
Fixed total particle number $N_0$ density: $\rho_0 = N_0/L$




$$\mathbb{W}_s = \sum_{k=1}^{L-1} \left[ e^{-s} (\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+) - \hat{n}_k (1 - \hat{n}_{k+1}) - (1 - \hat{n}_k) \hat{n}_{k+1} \right]$$

$$= \frac{L-1}{2} - \frac{e^{-s}}{2} \mathbb{H}_\Delta$$

$$\mathbb{H}_\Delta = - \sum_{k=1}^{L-1} \left[ \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \right] \qquad \text{with} \qquad \boxed{\Delta = e^s}$$

# Classical/Quantum dictionary

| **SSEP** | **Quantum Spin Chain** |
|---|---|
| local occupation number $n_k$ $(1 \leq k \leq L)$<br>$n_k = 0, 1 \equiv \circ, \bullet$ | local spin $\sigma_k^z$ $(1 \leq k \leq L)$<br>$\sigma_k^z = 1, -1 \equiv \uparrow, \downarrow$ |
| (fixed) total occupation $N_0 \equiv \rho_0 L$ | (fixed) total magnetization $M \equiv m_0 L$ |
| (mesoscopic) density $\rho(x)$ $(0 \leq x \leq 1)$ | (mesoscopic) magnet. $m(x)$ $(0 \leq x \leq 1)$ |

# Classical/Quantum dictionary

| **SSEP** | **Quantum Spin Chain** |
|---|---|
| local occupation number $n_k$ $(1 \le k \le L)$ <br> $n_k = 0, 1 \equiv \circ, \bullet$ | local spin $\sigma_k^z$ $(1 \le k \le L)$ <br> $\sigma_k^z = 1, -1 \equiv \uparrow, \downarrow$ |
| (fixed) total occupation $N_0 \equiv \rho_0 L$ | (fixed) total magnetization $M \equiv m_0 L$ |
| (mesoscopic) density $\rho(x)$ $(0 \le x \le 1)$ | (mesoscopic) magnet. $m(x)$ $(0 \le x \le 1)$ |
| evolution operator <br> $\mathbb{W}_s = \frac{L-1}{2} - \frac{e^{-s}}{2} \mathbb{H}_\Delta$ | ferromagnetic XXZ Hamiltonian $(J_{xy} = -1)$ <br> $\mathbb{H}_\Delta = \sum_{k=1}^{L-1} \Big[ J_{xy}\big(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y\big) + J_z \sigma_k^z \sigma_{k+1}^z \Big]$ <br> $= -\sum_{k=1}^{L-1} \Big[ \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \Big]$ |
| counting factor $\Delta = e^s$ of the activity $K$ | anisotropy $\Delta = -J_z$ along direction Z |
| cumulant generating function <br> $\psi(s) = \max \mathrm{Sp}\, \mathbb{W}_s = \frac{L-1}{2} - \frac{e^{-s}}{2} E_L(s)$ | ground state energy <br> $E_L(s) = \min \mathrm{Sp}\, \mathbb{H}_\Delta$ |

# Bethe Ansatz

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

# Bethe Ansatz

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Coordinate Bethe Ansatz: Integrability known from long ; difficulty: $L \to \infty$

- eigenvector of components

$$\sum_{\mathcal{P}} \mathcal{A}(\mathcal{P}) \prod_{i=1}^{N_0} \left[\zeta_{\mathcal{P}(i)}\right]^{x_i}$$

- eigenvalue

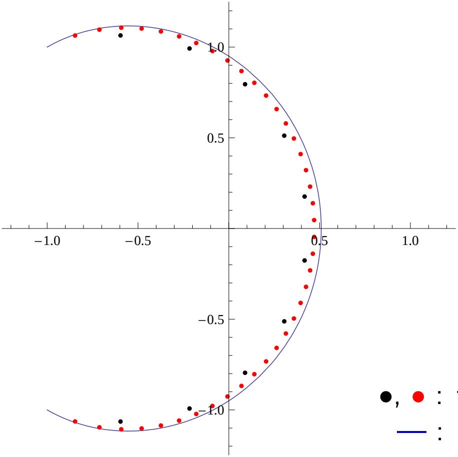
$$\psi(s) = -2N_0 + e^{-s}\left[\zeta_1 + \ldots + \zeta_{N_0}\right] - e^{-s}\left[\frac{1}{\zeta_1} + \ldots + \frac{1}{\zeta_{N_0}}\right]$$

- Bethe equations

$$\zeta_i^L = \prod_{\substack{j=1 \\ j\neq i}}^{N_0} \left[-\frac{1 - 2\mathrm{e}^{s}\zeta_i + \zeta_i\zeta_j}{1 - 2\mathrm{e}^{s}\zeta_j + \zeta_i\zeta_j}\right]$$

# Bethe Ansatz

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]




●, ● : finite-size solution
— : infinite-size limit

Repartition of Bethe roots in the complex plane

# Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

- large deviation function

$$\psi(s) = \underbrace{-2L\rho_0(1-\rho_0)s}_{\text{order } 0} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} + \dots \quad \text{with} \quad u = L^2\rho_0(1-\rho_0)s$$

- **universal function** (singular in $u = \frac{\pi^2}{2}$)

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

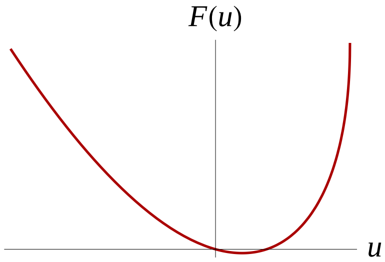
# Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

- large deviation function

$$\psi(s) = \underbrace{-2L\rho_0(1-\rho_0)s}_{\text{order } 0} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} + \dots \quad \text{with} \quad u = L^2\rho_0(1-\rho_0)s$$

- **universal function** (singular in $u = \frac{\pi^2}{2}$)



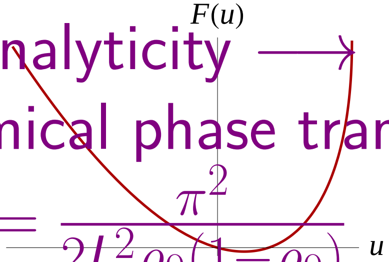
# Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

- large deviation function

$$\psi(s) = \underbrace{-2L\rho_0(1-\rho_0)s}_{\text{order } 0} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} + \dots \quad \text{with} \quad u = L^2\rho_0(1-\rho_0)s$$

- **universal function** (singular in $u = \frac{\pi^2}{2}$)




non-analyticity $\longrightarrow$

dynamical phase transition

at $s_c = \dfrac{\pi^2}{2L^2\rho_0(1-\rho_0)}$

# Macroscopic limit

a way to derive MFT

## A reminder: propagator in quantum mechanics

$$\langle \text{final} | e^{it\mathbb{H}} | \text{initial} \rangle$$

# Macroscopic limit

a way to derive MFT

## A reminder: propagator in quantum mechanics

$$\langle \text{final}|e^{it\mathbb{H}}|\text{initial}\rangle = \int dz_1 \ldots dz_n \langle \text{final}|e^{i\Delta t\mathbb{H}}|\underline{z_n}\rangle\langle\underline{z_{n-1}}|e^{i\Delta t\mathbb{H}}|\underline{z_{n-2}}\rangle \ldots$$
$$\ldots \langle\underline{z_1}|e^{i\Delta t\mathbb{H}}|\text{initial}\rangle$$
$$= \int \mathcal{D}p\mathcal{D}q \ \exp\{i\tfrac{1}{\hbar}\underbrace{\mathcal{S}[p,q]}_{\text{action}}\}$$

$p = p(x,t)$ and $q = q(x,t)$
are generically space- & time-dependent **fields**.

"semi-classical limit" recovered in the large $\frac{1}{\hbar}$ limit [saddle-point]

# Macroscopic limit [Tailleur, Kurchan, VL, JPA **41** 505001]

For exclusion processes

Using $SU(2)$ coherent states:

$$\langle \rho_{\mathsf{f}} | e^{t\mathbb{W}} | \rho_{\mathsf{i}} \rangle = \int_{\rho(0)=\rho_{\mathsf{i}}}^{\rho(t)=\rho_{\mathsf{f}}} \mathcal{D}\rho\mathcal{D}\hat{\rho} \; \exp\{ L \underbrace{\mathcal{S}[\hat{\rho},\rho]}_{\text{action}} \}$$

# Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

For exclusion processes

Using $SU(2)$ coherent states:

$$\langle \rho_{\mathsf{f}} | e^{t\mathbb{W}} | \rho_{\mathsf{i}} \rangle = \int_{\rho(0)=\rho_{\mathsf{i}}}^{\rho(t)=\rho_{\mathsf{f}}} \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L \underbrace{\mathcal{S}[\hat{\rho},\rho]}_{\text{action}}\}$$

$$\langle e^{-sK} \rangle \sim \langle \rho_{\mathsf{f}} | e^{t\mathbb{W}_s} | \rho_{\mathsf{i}} \rangle = \int_{\rho(0)=\rho_{\mathsf{i}}}^{\rho(t)=\rho_{\mathsf{f}}} \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L \underbrace{\mathcal{S}_s[\hat{\rho},\rho]}_{\text{action}}\}$$

Again: use saddle-point to handle the large $L$ limit.

# Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]

For exclusion processes

Same $\mathcal{S}_s[\hat\rho,\rho]$ as the MSR action of the Langevin evolution:

$$\partial_t\rho(x,t) = -\partial_x\big[-\partial_x\rho(x,t)+\xi(x,t)\big]$$

$$\langle\xi(x,t)\xi(x',t')\rangle = \frac{1}{L}\rho(x,t)\big(1-\rho(x,t)\big)\,\delta(x'-x)\delta(t'-t)$$

One recovers the action of fluctuating hydrodynamics

[Spohn; Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim]

# $\psi(s)$: again [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

$$\frac{1}{Lt}\langle Q\rangle \propto D(\rho) \qquad \text{(Fourier's law)}$$

$$\frac{1}{Lt}\langle Q^2\rangle_c = \sigma(\rho) \qquad \text{(For the SSEP, } \sigma(\rho)=\rho(1-\rho)\text{)}$$

# $\psi(s)$: again [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions

More general fluctuating hydrodynamics

$$\frac{1}{Lt}\langle Q\rangle \propto D(\rho) \qquad \text{(Fourier's law)}$$

$$\frac{1}{Lt}\langle Q^2\rangle_c = \sigma(\rho) \qquad \text{(For the SSEP, } \sigma(\rho) = \rho(1-\rho)\text{)}$$

Saddle point evaluation

$$\langle e^{-sK}\rangle \sim \int \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L\,\mathcal{S}_s[\hat{\rho},\rho]\}$$

# $\psi(s)$: again [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

$$\frac{1}{Lt}\langle Q\rangle \propto D(\rho) \qquad \text{(Fourier's law)}$$

$$\frac{1}{Lt}\langle Q^2\rangle_c = \sigma(\rho) \qquad \text{(For the SSEP, } \sigma(\rho)=\rho(1-\rho)\text{)}$$

Saddle point evaluation

$$\langle e^{-sK}\rangle \sim \int \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L\,\mathcal{S}_s[\hat{\rho},\rho]\}$$

Large deviation function [assuming **uniform** profile $\rho(x)=\rho$]

$$\psi(s) = \underbrace{-s\frac{\langle K\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\int \text{ of quadratic fluctuations}} \quad \text{with} \quad u = L^2 s\frac{\sigma(\rho_0)\sigma''(\rho_0)}{8D^2}$$

Correspondence between
the (Gaussian) integration of small fluctuations
AND
discreteness of Bethe root repartition.

More general?

Correspondence between
the (Gaussian) integration of small fluctuations
AND
discreteness of Bethe root repartition.

More general?

Repartition of Bethe roots for $s > s_c$?

Correspondence between
the (Gaussian) integration of small fluctuations
AND
discreteness of Bethe root repartition.

More general?

Repartition of Bethe roots for $s > s_c$?

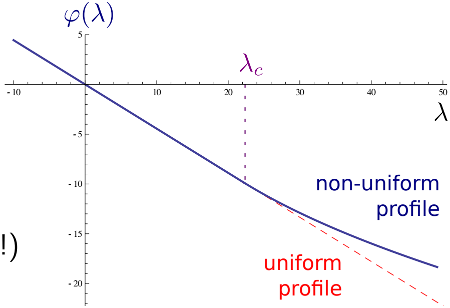
Fluctuating hydrodynamics for quantum chains?

# Dynamical phase transition [VL, Garrahan, van Wijland, JPA **45** 175001]

Rescaling of the large deviation function [singularity at $\lambda_c > 0$ as $L \to \infty$ ]

$$\varphi(\lambda) = \lim_{L\to\infty} L\psi(\lambda/L^2)$$

Using the correct *non-uniform* saddle-point profile for $\lambda > \lambda_c$




$\lambda_c = \frac{\pi^2}{\sigma(\rho_0)}$
(can be large!)

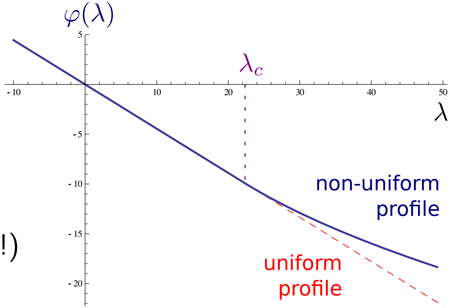
# Dynamical phase transition [VL, Garrahan, van Wijland, JPA **45** 175001]

Rescaling of the large deviation function [singularity at $\lambda_c > 0$ as $L \to \infty$ ]

$$\varphi(\lambda) = \lim_{L\to\infty} L\psi(\lambda/L^2)$$

Using the correct *non-uniform* saddle-point profile for $\lambda > \lambda_c$

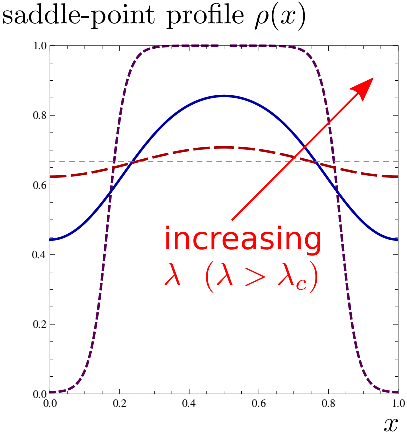
$\lambda_c = \frac{\pi^2}{\sigma(\rho_0)}$
(can be large!)




see also: for LDF of $Q$ [Bodineau, Derrida, PRE **78** 021122]
phase transition in WASEP for large dev. (**non-stationary** profile) [Jona-Lasinio *et al.*]
generic criterion for instability

# Dynamical phase transition [VL, Garrahan, van Wijland, JPA **45** 175001]

Optimal



| SSEP | Quantum Spin Chain |
|---|---|
| local occupation number $n_k$ $(1 \leq k \leq L)$ | local spin $\sigma_k^z$ $(1 \leq k \leq L)$ |
| $n_k = 0, 1 \equiv \circ, \bullet$ | $\sigma_k^z = 1, -1 \equiv \uparrow, \downarrow$ |
| (fixed) total occupation $N_0 \equiv \rho_0 L$ | (fixed) total magnetization $M \equiv m_0 L$ |
| (mesoscopic) density $\rho(x)$ $(0 \leq x \leq 1)$ | (mesoscopic) magnet. $m(x)$ $(0 \leq x \leq 1)$ |
| evolution operator $\mathbb{W}_s$ | ferromagnetic XXZ Hamiltonian $\mathbb{H}_\Delta$ |
| cumulant generating function $\psi(s)$ | ground state energy $E_L(s)$ |

| SSEP | Quantum Spin Chain |
|---|---|
| local occupation number $n_k$ $(1 \leq k \leq L)$<br>$n_k = 0, 1 \equiv \circ, \bullet$ | local spin $\sigma_k^z$ $(1 \leq k \leq L)$<br>$\sigma_k^z = 1, -1 \equiv \uparrow, \downarrow$ |
| (fixed) total occupation $N_0 \equiv \rho_0 L$ | (fixed) total magnetization $M \equiv m_0 L$ |
| (mesoscopic) density $\rho(x)$ $(0 \leq x \leq 1)$ | (mesoscopic) magnet. $m(x)$ $(0 \leq x \leq 1)$ |
| evolution operator $\mathbb{W}_s$<br>cumulant generating function $\psi(s)$ | ferromagnetic XXZ Hamiltonian $\mathbb{H}_\Delta$<br>ground state energy $E_L(s)$ |
| minimal activity phase $(s \to +\infty)$<br>$\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\circ\circ\circ\circ\circ$ | Ising ferromagnetic order $(\Delta \to +\infty)$<br>$\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow$ |
| maximal activity phase $(s \to -\infty)$<br>$\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ$<br>& $\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet$ | XY degenerate groundstate $(\Delta = 0)$<br>$\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$<br>(in fact, superp. of $e^{i\theta}\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\ldots$) |
| time $t$ (steady state: $t \to +\infty$) | inverse temp. $\beta$ (zero-temp. limit: $\beta \to +\infty$) |
| dynamical partition function<br>$\langle e^{-sK} \rangle \simeq \mathrm{Tr}\, e^{t\mathbb{W}_s}$ | partition function<br>$Z_\beta^{\mathrm{XXZ}}(\Delta) = \mathrm{Tr}\, e^{-\beta\,\mathbb{H}_\Delta}$ |

# Sketch of derivation [VL, Garrahan, van Wijland, JPA **45** 175001]

Saddle-point equations for the profile $\rho(x)$ take the form

$$\big(\partial_x \rho(x)\big)^2 + E_P\big(\rho(x)\big) = 0$$

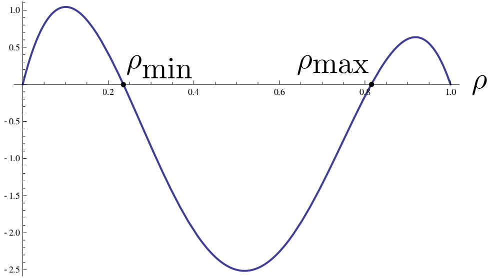
# Sketch of derivation [VL, Garrahan, van Wijland, JPA **45** 175001]

Saddle-point equations for the profile $\rho(x)$ take the form

$$\big(\partial_x \rho(x)\big)^2 + E_P\big(\rho(x)\big) = 0$$

Motion in "time" $x$ of a particle of "position" $\rho$ in a

"Potential energy" $E_P(\rho)$



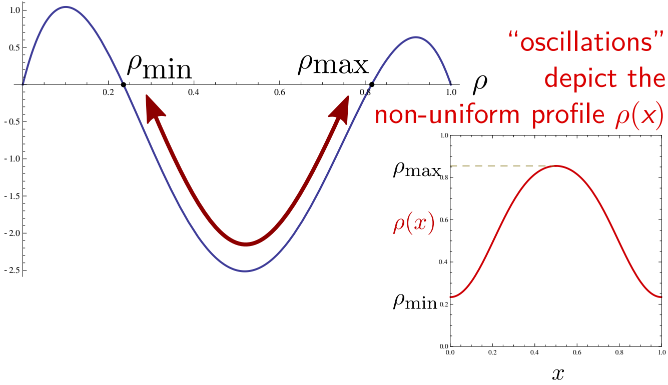
# Sketch of derivation [VL, Garrahan, van Wijland, JPA **45** 175001]

Saddle-point equations for the profile $\rho(x)$ take the form

$$\big(\partial_x \rho(x)\big)^2 + E_P\big(\rho(x)\big) = 0$$

Motion in "time" $x$ of a particle of "position" $\rho$ in a

"Potential energy" $E_P(\rho)$

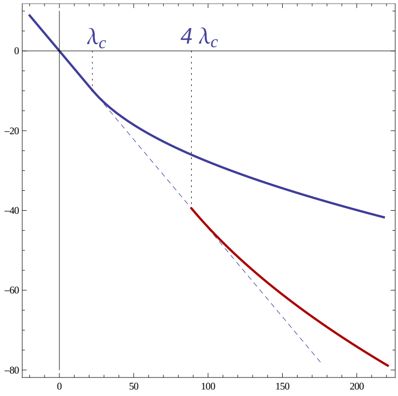


# Excitations

[Cheneau, VL, *work in progress*]

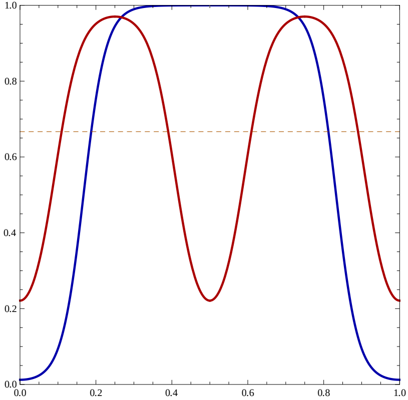
What about solutions with *more than one* kink+anti-kink?

$\varphi(\lambda)$




$\lambda$

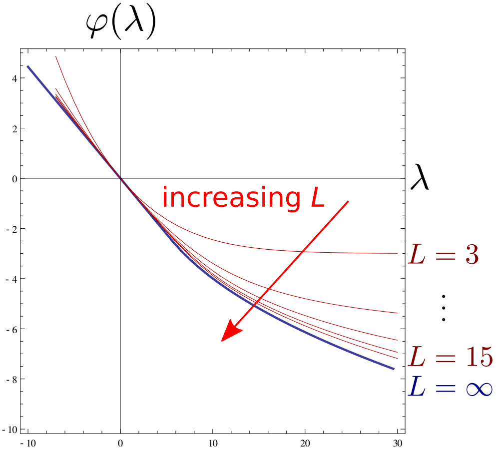
corresponding profiles $\rho(x)$




$x$

# Small sizes: the ground state

Aim: experimental realizations with cold atoms
$\rightarrow$ non-periodic (but isolated, 1D) system
$\rightarrow$ smaller sizes & finite-temperature & excited state

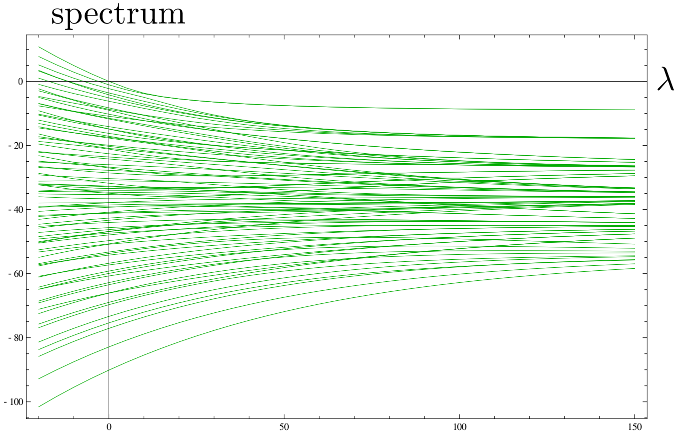


# Small sizes: the full spectrum [preliminary!]

$L = 9$ sites
$N_0 = 3$ particles

# Small sizes: the full spectrum [preliminary!]

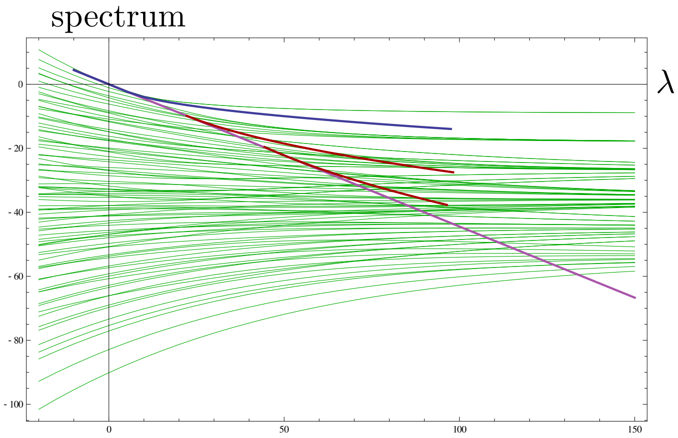
$L = 9$ sites
$N_0 = 3$ particles



# Small sizes: the full spectrum [preliminary!]

$L = 9$ sites
$N_0 = 3$ particles

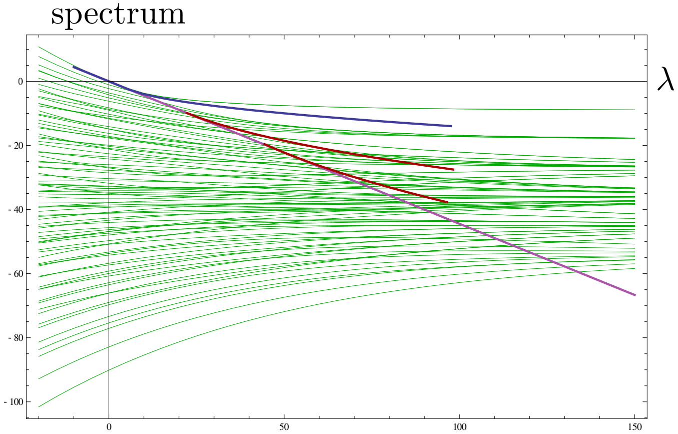
infinite-size ground state
infinite-size excited states



# Small sizes: the full spectrum [preliminary!]

$L = 9$ sites
$N_0 = 3$ particles

infinite-size ground state
infinite-size excited states




gathering(?) of microscopic eigenvalues $\longrightarrow$ macroscopic ($L = \infty$) states

# Summary

Microscopic approach:

- ⋆ operator formalism
- ⋆ XXZ spin chain
- ⋆ Bethe Ansatz

Macroscopic approach:

- ⋆ MFT, saddle-point method, dynamical phase transition

# Summary

## Microscopic approach:

- ⋆ operator formalism ⋆ XXZ spin chain ⋆ Bethe Ansatz

## Macroscopic approach:

- ⋆ MFT, saddle-point method, dynamical phase transition

## Questions:

- ⋆ Finite-size crossover around a quantum phase transition? Between:
  - · Luttinger Liquid ($s \to -\infty$)
  - · Phase-separated ferromagnet ($s \to +\infty$)
- ⋆ Across the transition: continuum spectrum $\to$ gaped spectrum?
- ⋆ XXZ transition not at $\Delta = 1$ but at $\Delta = 1 + \mathcal{O}(L^{-2})$
- ⋆ Are scaling exponents/functions known? Are they interesting?
- ⋆ Hydrodynamics approaches for quantum questions?
- ⋆ Non-Hermitian operators $\longleftrightarrow$ dissipation in Lindblad?

# Thank you for your attention!

References:


- ⋆ Marc Cheneau, Vivien Lecomte et al.
  *work in progress* (2014)
- ⋆ Vivien Lecomte, Juan P. Garrahan, Frédéric van Wijland
  J. Phys. A **45** 175001 (2012)
- ⋆ Vivien Lecomte, Alberto Imparato, Frédéric van Wijland
  PTPS **184** 276 (2010)
- ⋆ Cécile Appert-Rolland, Bernard Derrida, Vivien Lecomte, Frédéric van Wijland
  Phys. Rev. E **78** 021122 (2008)