Partial exam

Contents

Reminder

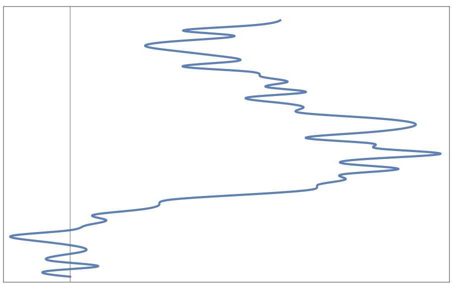
We say that a Brownian motion W_x is a Brownian motion of coordinate x and diffusion constant D when $W_0 = 0$ and

$$\langle (W_x - W_{x'})^2 \rangle = D|x - x'| \tag{1}$$

This 2-point correlation function fully characterizes the distribution of W_x. Explain why.

1 Fluctuations for a Kardar-Parisi-Zhang interface (in the directed polymer approach)

Consider a one-dimensional 'interface' or 'polymer': a trajectory $y(t)$ starting at the origin ($y(0) = 0$).



One is interested in the behaviour of the average variance at large scale t

$$B(t) = \langle y(t)^2 \rangle \tag{2}$$

It depends on the distribution of the trajectory. (Here and below we assume a symmetric distribution: $\langle y(t) \rangle = 0$).

1.1 Without disorder

A good model for an continuous interface living in an environment without disorder is to consider that $y(t)$ is a Brownian motion of coordinate t and diffusion coefficient equal to 1.

1. Determine $B(t)$. Find the *roughness exponent* ζ (defined by $B(t) \sim Ct^{2\zeta}$ as $t \to \infty$, where C is a constant).

2. What is the distribution $P(y, t)$ of the "arrival point" y of the interface at scale t?

1.2 With disorder

To implement the influence of disorder in the environment seen by the interface, one might justify that a good model is given by the following distribution of the 'arrival point' y of the interface at fixed scale t:

$$P_V(y,t) \propto e^{-\frac{1}{2}\frac{y^2}{t}+V(y)} \tag{3}$$

Here $V(y)$ is a Brownian motion of coordinate y and diffusion constant D. It can be understood as a "disorder" seen by the extremity y of the interface at scale t

1. When $D=0$, does V play a role? Do we recover the model "without disorder" of §.1.1?
2. Justify in details that the roughness, averaged over disorder, writes

$$B(t) = \left\langle \frac{\displaystyle\int dy\, y^2\, e^{-\frac{1}{2}\frac{y^2}{t}+V(y)}}{\displaystyle\int dy\, e^{-\frac{1}{2}\frac{y^2}{t}+V(y)}} \right\rangle_V \tag{4}$$

   where $\langle ... \rangle_V$ represents the average over V

3. Consider a generic Brownian motion W_x of coordinate x and diffusion constant D. Using the Reminder around (1), show that

$$\left\langle F\left[W_{a\bar{x}}\right]\right\rangle_W = \left\langle F\left[a^{1/2}\bar{W}_{\bar{x}}\right]\right\rangle_{\bar{W}} \tag{5}$$

   Where $\bar{W}_{\bar{x}}$ is a Brownian motion of coordinate $\bar{x}$ and diffusion constant D. In this expression F is a function (or "functional") which depends on many values of W_x (for instance $F[W_x] = \int dx W_x$)

4. Using the results of the two previous questions, perform in (4) the change of variable $y = t^z \bar{y}$ and find the exponent z such that (4) takes the form

$$B(t) = \left\langle \frac{\displaystyle\int d\bar{y}\, (t^z\bar{y})^2\, e^{t^\alpha\left[-\frac{1}{2}\bar{y}^2+\bar{V}(\bar{y})\right]}}{\displaystyle\int d\bar{y}\, e^{t^\alpha\left[-\frac{1}{2}\bar{y}^2+\bar{V}(\bar{y})\right]}} \right\rangle_{\bar{V}} \tag{6}$$

   where you have to determine the exponent α.

5. We admit that the following *saddle-point evaluation* is valid:

$$\int d\bar{y}\, f(\bar{y})\, e^{t^\alpha\left[-\frac{1}{2}\bar{y}^2+\bar{V}(\bar{y})\right]} \sim f(\bar{y}^\star)\, e^{t^\alpha\left[-\frac{1}{2}(\bar{y}^\star)^2+\bar{V}(\bar{y}^\star)\right]} \quad \text{as } t\to\infty \tag{7}$$

   where $\bar{y}^\star$ is the point where $t^\alpha\left[-\frac{1}{2}\bar{y}^2+\bar{V}(\bar{y})\right]$ reaches its maximum. Does $\bar{y}^\star$ depends on t? Does $\bar{y}^\star$ depends on V? We admit that $\bar{y}^\star$ does not depends on f in our case.

6. Use this result in (6) so as to obtain the roughness exponent ζ in

$$B(t) \sim Ct^{2\zeta} \quad \text{as } t\to\infty, \tag{8}$$

   where C is a constant. How does ζ compare to the case without disorder? Provide a physical interpretation.

7. [**Long**] Following a similar route (*i.e.* using the Reminder around (1) so as to express $B(t)$ in a form similar to (6), but now with $\bar{V}$ of diffusion constant 1, and D appearing elsewhere, thanks to a rescaling of the form $y = At^z\bar{y}$, with A a well-chosen constant), find how the constant C in (8) depends on D.

2 Large deviations in Markov chains

We consider a finite set of configurations $\mathcal{C}$ and a continuous time Markov chain described by transition rates $W(\mathcal{C} \to \mathcal{C}')$.

2.1 Statistics of the activity K

We focus on histories of duration t and call K the number of events (or "jumps between configurations") during the time window $[0, t]$. We are interested in the statistics of K.

1. Justify that
$$\partial_t P(\mathcal{C}, K, t) = \sum_{\mathcal{C}'} W(\mathcal{C}' \to \mathcal{C}) P(\mathcal{C}', K-1, t) \;-\; r(\mathcal{C}) P(\mathcal{C}, K, t) \tag{9}$$

2. From this, show in details that
$$\partial_t \langle K \rangle = \langle r(\mathcal{C}) \rangle \tag{10}$$
where you have to define the averages on both sides of this equality.

   [Hint: multiply (9) by K and sum over K ; note also that $\sum_K P(\mathcal{C}, K, t) = P(\mathcal{C}, t)$ and explain why this equality holds.]

3. We introduce the Laplace transform
$$\hat{P}(\mathcal{C}, s, t) = \sum_K e^{-sK} P(\mathcal{C}, K, t) \tag{11}$$
Show that
$$\left\langle e^{-sK} \right\rangle = \sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t) \tag{12}$$
where you have to specify what is the average on the left hand side.

4. We thus see that the study of the time behaviour of $\hat{P}(\mathcal{C}, s, t)$ gives access to $Z(s, t)$ defined as $Z(s, t) = \left\langle e^{-sK} \right\rangle$. We introduce the "cumulant generating function" $\psi(s)$ as
$$\psi(s) = \frac{1}{t} \log Z(s, t) \tag{13}$$
Show that $\psi(0) = 0$, $\psi'(0) = -\frac{1}{t}\langle K \rangle$, and $\psi''(0) = \frac{1}{t}[\langle K^2 \rangle - \langle K \rangle^2]$.

   The function $\psi(s)$ is called the "cumulant generating function" (its successive derivatives in 0 provide the cumulants of K). It describes the statistics of K.

5. From (9) and (11), write the equation of evolution of $\hat{P}(\mathcal{C}, s, t)$ as
$$\partial_t \hat{P}(\mathcal{C}, s, t) = \dots \tag{14}$$

6. Introducing the vector $|\hat{P}(s, t)\rangle = \sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t) |\mathcal{C}\rangle$ whose components are the $\hat{P}(\mathcal{C}, s, t)$, rewrite the previous evolution equation in a linear form
$$\partial_t |\hat{P}(s, t)\rangle = \mathbb{W}(s) |\hat{P}(s, t)\rangle \tag{15}$$
and show in details that the components of the *deformed* evolution operator $\mathbb{W}(s)$ are
$$\mathbb{W}(s)_{\mathcal{C}\mathcal{C}'} = e^{-s} W(\mathcal{C}' \to \mathcal{C}) - r(\mathcal{C}) \delta_{\mathcal{C}\mathcal{C}'} \tag{16}$$

7. Does $\mathbb{W}(s)$ preserve probability?

8. Show that $Z(s, t) = \langle \mathbf{1} | e^{t\mathbb{W}(s)} | P_0 \rangle$ where P_0 is the initial distribution and $\langle \mathbf{1}| = \sum_{\mathcal{C}} \langle \mathcal{C}|$.

9. Show in details that in the large time limit $\psi(s)$ is the maximal eigenvalue of $\mathbb{W}(s)$.

2.2 Statistics of the time-integrated escape rate R

We are now interested in the statistics of the time-integrated escape rate R, on a time window $[0, t]$

$$R(t) = \int_0^t d\tau\, r(\mathcal{C}(\tau)) \tag{17}$$

where $\mathcal{C}(\tau)$ represents the configuration at time τ.

1. [**Long** – you can admit the result.] Using the explicit expression of the probability of a trajectory (obtained by time-ordered exponential method), show that

$$\langle e^{-\sigma R} \rangle = \langle \mathbf{1} | e^{t\bar{\mathbb{W}}(\sigma)} | P_0 \rangle \tag{18}$$

   where $\bar{\mathbb{W}}(\sigma)$ is a matrix of components

$$\bar{\mathbb{W}}(\sigma)_{\mathcal{C}\mathcal{C}'} = W(\mathcal{C}' \to \mathcal{C}) - (1+\sigma) r(\mathcal{C}) \delta_{\mathcal{C}\mathcal{C}'} \tag{19}$$

   and P_0 the initial distribution.

2. Using this result, explain why $\bar{\psi}(\sigma)$, the cumulant generating function for R,

$$\bar{\psi}(\sigma) = \frac{1}{t} \log \langle e^{-\sigma R} \rangle \tag{20}$$

   is given by the largest eigenvalue of $\bar{\mathbb{W}}(\sigma)$.

3. Find a relation of the form $\mathbb{W}(s) = a(s)\bar{\mathbb{W}}(b(s))$ where $a(s)$ and $b(s)$ are two functions of s that you have to determine. Deduce from it a relation of the form $\psi(s) = (...)\bar{\psi}(...)$

4. What does this mean for the relation between the statistics of K and the statistics of R? Relate the 2 first cumulants of K and the 2 first cumulants of R.