Dynamical symmetry breaking and phase transitions in driven diffusive systems

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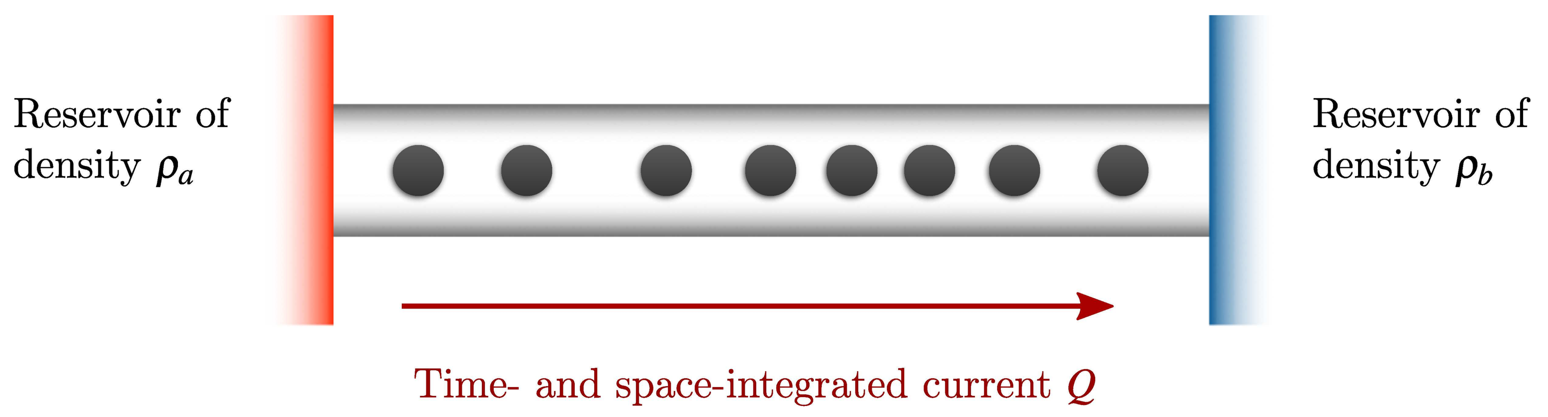
Abstract

We study the probability distribution of a **time-averaged current** flowing through a 1D diffusive system, connected to a pair of reservoirs at its two ends. Sufficient conditions for the occurrence of a host of possible dynamical phase transitions both in and out of equilibrium are derived. These transitions manifest themselves as singularities in **large deviation** functions, resulting in enhanced current fluctuations. Microscopic models which implement each of the scenarios are presented, with possible experimental realisations. Depending on the model, the singularity is associated either with a particle-hole **symmetry breaking**, which leads to a continuous transition, or in the absence of the symmetry with a **first-order phase transition**. An exact **Landau theory** which captures the different singular behaviours is derived.

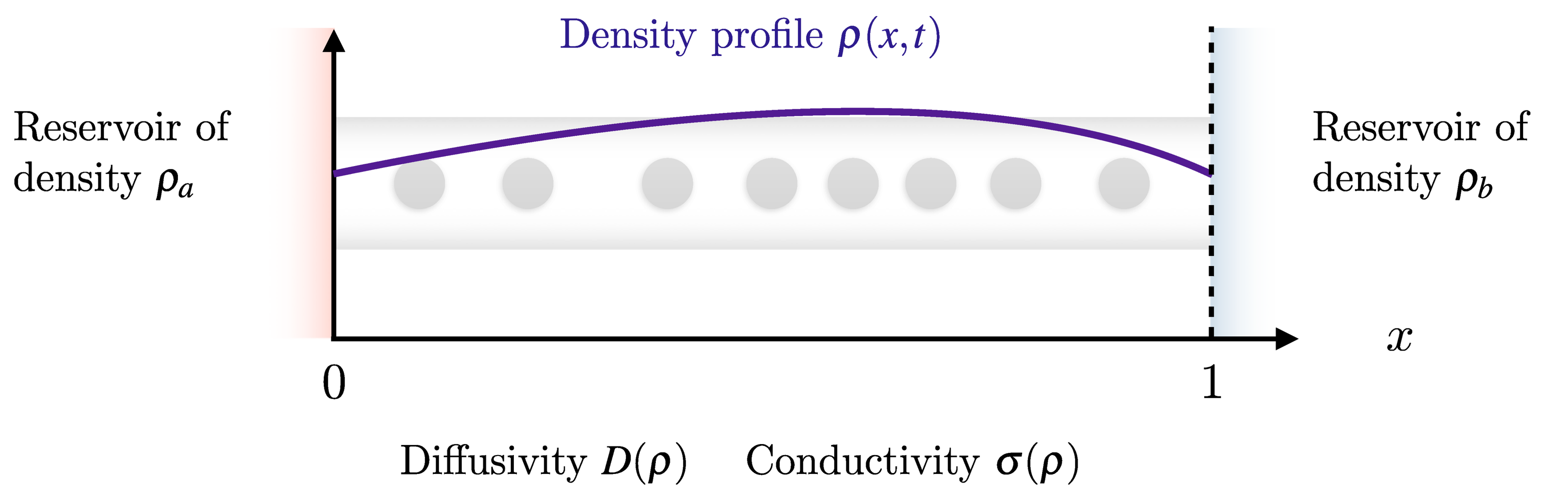
Current distribution in and out of equilibrium

- **Microscopic description**: particles interact in a 1D channel connected to reservoirs



- **Observables**:
  - ⋆ Current Q on a time window $[0,t_f]$: Q = (# jumps to the right) − (# jumps to the left)
  - ⋆ Activity K on a time-window $[0,t_f]$: K = (# jumps to the right) + (# jumps to the left)
- **Macroscopic description**: particle density field $\rho(x,t)$, current field $j(x,t)$

Integrated current: $Q = \int_0^1 dx \int_0^{t_f} dt\, j(x,t)$



Large deviations

- **Mean value**: $\frac{1}{t_f}Q \to \bar{j}$ as $t_f \to \infty$.
- What about **fluctuations**? As $t_f \to \infty$, the distribution of $\frac{1}{t_f}Q$ obeys a *large-deviation principle*

$$\mathrm{Prob}\big(\tfrac{1}{t_f}Q = J\big) \sim e^{-t_f \varphi_L(J)} \qquad \text{where } \varphi_L(J) \text{ is the ‘rate function’}$$

- **Canonical** instead of **micro-canonical** approach: the *moment-generating function*

$$\left\langle e^{-sQ}\right\rangle \sim e^{t_f \psi_L(s)}$$

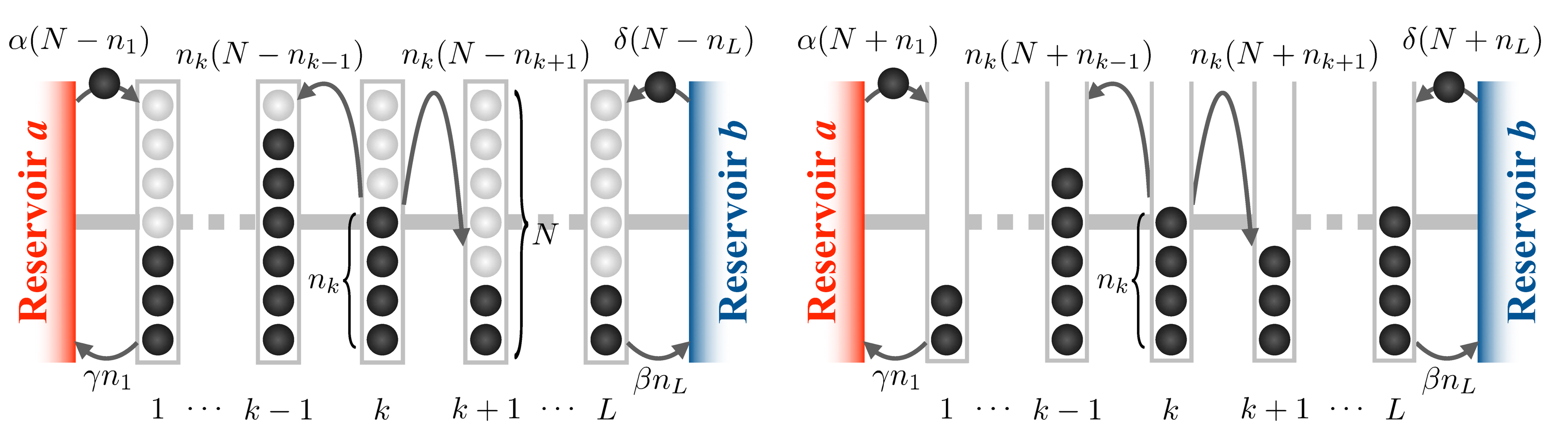
- Analogy: $\varphi_L(J)$ plays the role of a dynamical **entropy**, $\psi_L(s)$ a dynamical **free energy**.
- Connection through a dynamical **change of ensembles**:
  - ⋆ ‘s-ensemble’: histories biased by $\langle e^{-sQ}\rangle$
  - ⋆ Typical value $j_L(s)$ of $\frac{1}{t_f}Q$ in the s-ensemble:

$$j_L(s) = \lim_{t_f\to\infty} \frac{1}{t_f}\frac{\left\langle Qe^{-sQ}\right\rangle}{\langle e^{-sQ}\rangle} = -\psi_L'(s)$$

  - ⋆ In other words: a given s in the s-ensemble corresponds to histories of the non-biased system with an atypical current of value $j_L(s) = -\psi_L'(s)$. In particular: $s = 0 \Leftrightarrow j_L(s) = \bar{j}$.
  - ⋆ **Legendre transform**:

$$\psi_L(s) = -\min_j \left\{sj + \varphi_L(j)\right\} \quad \text{(the minimum is reached in } j = j_L(s))$$
$$\varphi_L(j) = -\min_s \left\{sj + \psi_L(s)\right\} \quad \text{(min reached in } s = s_L(j) \text{ such that } j = j_L(s_L(j)))$$

The example of large-N models



Simple Partial Exclusion Process (SPEP) Simple Inclusion Process (SIP)

Figure: Jump rates (for sites with occupancies between 0 and N per site).

Rescaling of the large deviation functions: $\psi_L(s = \lambda/N) = \Psi_L(\lambda)$ $\qquad \varphi_L(j = J/N) = \Phi_L(J)$.

Macroscopic Fluctuation Theory [1] in a nut-shell

- **Continuity equation**: $\partial_t\rho(x,t) + \partial_x j(x,t) = 0$, which is in fact...
- ... a **Langevin equation** for the **fluctuating** current: $j(x,t) = -D(\rho)\partial_x\rho + \sigma(\rho)E + \sqrt{\sigma(\rho)}\eta(x,t)$
- Gaussian small noise: $\langle\eta(x,t)\eta(x',t')\rangle = L^{-1}\delta(x-x')\delta(t-t')$ with $L \gg 1$ the number of lattice sites.
- Corresponding Martin-Siggia-Rose-Jansen-de Dominicis representation for $\Psi(\lambda) = \frac{1}{L}\Psi_L(\lambda/L)$:

$$\Psi(\lambda) = \lim_{t_f\to\infty}\frac{1}{Lt_f}\log\int \mathcal{D}\rho\mathcal{D}\hat{\rho}\, e^{-L\int_0^{t_f}dt\int_0^1 dx\,[\hat{\rho}\partial_t\rho - H(\rho,\hat{\rho})]}$$

- **Hamiltonian**: $H(\rho,\hat{\rho}) \equiv -D(\rho)(\partial_x\rho)(\partial_x\hat{\rho}) + \frac{\sigma(\rho)}{2}(\partial_x\hat{\rho})(2E + \partial_x\hat{\rho})$
- Large-L limit (small noise) yields **saddle-point** equations of the **optimal** $\rho(x,t)$.

Known results

For processes related to the Simple Exclusion Process (constant $D(\rho)$, quadratic $\sigma(\rho)$):

[2] **Additivity principle**: the optimal profile is time-independent for not too large deviations.
[3] *Periodic* boundary conditions: the optimal profile is a **travelling wave** for large enough λ.
[4] *Periodic* boundary conditions: for K, the opt. prof. is **stationary but non-uniform** for (idem).
[5] *Periodic* 2D system: on top of the previous 2nd-order transitions, 1st also exist.

The approach we followed [6]: *construct* a Landau theory

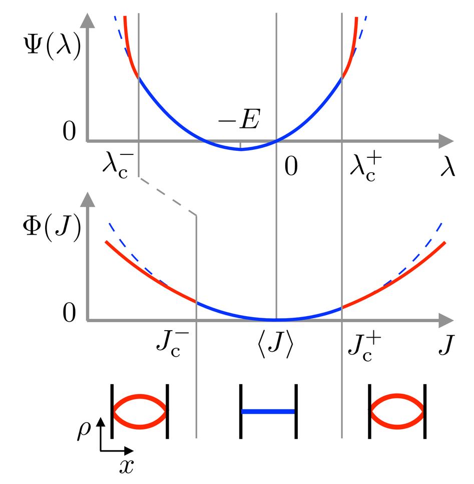
Method (for *any functions* $D(\rho)$ and $\sigma(\rho)$):

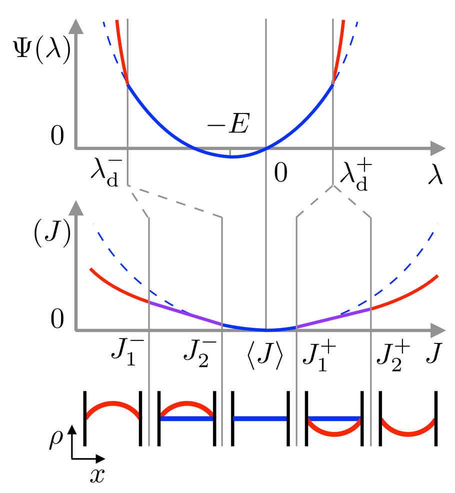
- Assume that $\sigma(\rho)$ has an extremum in $\rho = \bar{\rho}$, *i.e.* $\sigma'(\bar{\rho})$.
- Take boundary conditions with **reservoir** of densities $\rho_a = \rho_b = \bar{\rho}$.
- Check that the flat profile $\rho(x) = \bar{\rho}$ is an optimal profile.
- Observe that under certain conditions (and for $|\lambda - \lambda_0| > \lambda_c$), the **non-uniform** profile

$$\rho_m(x) = \bar{\rho} + m\sin\pi x + O(m)$$

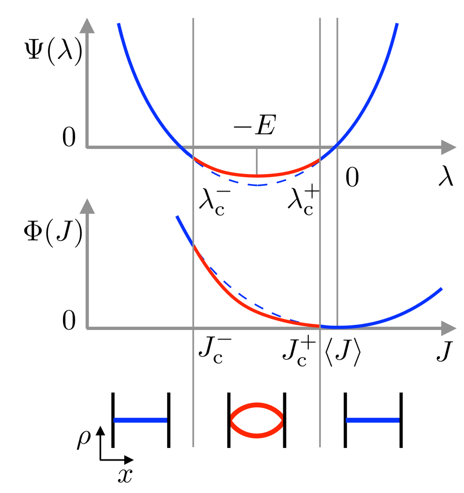
  is also an optimal profile, **better than the flat one**.
- Construct a **Landau theory** for the sole parameter m (instead of the full functional space). (This involves an explicit expansion of $\rho_m(x)$ up to order m^4.)
- Treat the case $\rho_{a,b} = \bar{\rho} \pm \delta\rho$ perturbatively in $\delta\rho \ll 1$. Idem for a bulk field E.
- Determine the criteria for the following **possible transitions**:

(*i*) **Second-order singularity for large current**:
- ⋆ the rescaled cumulant-generating function $\Psi(\lambda)$
- ⋆ the rescaled rate function $\Phi(J)$
- ⋆ the corresponding optimal profiles





(*ii*) **First-order singularity for large current.**



(*iii*) **Second-order singularity for small current.**

Remarks:

- Origin of the transition **very different** from the case of periodic boundary conditions.
- **Physical picture**:
  - ⋆ If $\sigma''(\bar{\rho}) > 0$: transition ⇐ competition btw diffusion (favours flat) & noise (fav. modulations)
  - ⋆ If $\sigma''(\bar{\rho}) < 0$: transition ⇐ competition btw diffusion (favours flat) & field E (fav. modulations)
- **Explicit system** realisations: Katz-Lebowitz-Spohn (KLS) model; a baby-KLS model; WASEP.

Open questions

- Spatially **discrete** systems (in the large-N limit): existence and characterisation of transitions.
- **Experimental** realisations? (Heat current in RC circuits [Ciliberto’s group]). **Quantum** analog?
- Non-uniform and **time-dependent** optimal solutions? (Observed numerically in [7].)

References

[1] Lorenzo Bertini, Alberto De Sole, Davide Gabrielli, Giovanni Jona-Lasinio, and Claudio Landim. Macroscopic fluctuation theory. *Rev. Mod. Phys.* **87**, 593 (2015).

[2] Thierry Bodineau and Bernard Derrida. Current Fluctuations in Nonequilibrium Diffusive Systems: An Additivity Principle. *Phys. Rev. Lett.* **92**, 180601 (2004).

[3] Thierry Bodineau and Bodineau Derrida. Distribution of current in nonequilibrium diffusive systems and phase transitions. *Phys. Rev. E* **72**, 066110 (2005).

[4] Cécile Appert-Rolland, Bernard Derrida, Vivien Lecomte, and Frédéric van Wijland. Universal cumulants of the current in diffusive systems on a ring. *Phys. Rev. E* **78**, 021122 (2008).

[5] Nicolás Tizón-Escamilla, Carlos Pérez-Espigares, Pedro L. Garrido, and Pablo I. Hurtado. Order and symmetry-breaking in the fluctuations of driven systems. *arXiv:1606.07507 [cond-mat]* (2016).

[6] Yongjoo Baek, Yariv Kafri, and Vivien Lecomte. Dynamical Symmetry Breaking and Phase Transitions in Driven Diffusive Systems. *Phys. Rev. Lett.* **118**, 030604 (2017).

[7] Ohad Shpielberg, Yaroslav Don, and Eric Akkermans. Numerical Study of Continuous and Discontinuous Dynamical Phase Transitions for Boundary Driven Systems. *arXiv:1612.07605 [cond-mat]* (2016).