Dynamical Phases in Stochastic Systems

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Abstract

On the macroscopic scale, and for most of their properties, systems in equilibrium can be described without prior knowledge of their dynamics. This is at variance with what occurs in out-of-equilibrium systems (with slow glassy dynamics, or in far from equilibrium steady-states) where the microscopic dynamics is the key to the systems' macroscopic features.

To probe aspects intrinsic to the dynamics of physical systems, we imported concepts of the theory of dynamical systems into the description of systems with Markov dynamics [1]. These consist in focusing on the various histories (and their fluctuations) that the systems may follow.

We will show on specific examples how these tools can shed light onto the "dynamical phases" of such systems, even when considering simple equilibrium processes.

Dynamical Complexity

We consider systems endowed with *continuous time* Markovian dynamics, of transition rates $W(\mathcal{C}\to\mathcal{C}')$ between different configurations. The probability $P(\mathcal{C},t)$ for the system to be in a configuration $\mathcal{C}$ obeys the master equation:

$$\partial_t P(\mathcal{C},t)=\sum_{\mathcal{C}'}\Big[W(\mathcal{C}'\to\mathcal{C})P(\mathcal{C}',t)-W(\mathcal{C}\to\mathcal{C}')P(\mathcal{C},t)\Big]$$

- In a state of probability distribution $P(\mathcal{C})$, the Gibbs-Boltzman entropy $S[P]$ embodies the (***static***) complexity of the system:

$$S[P]=-\sum_{\mathcal{C}}P(\mathcal{C})\ln P(\mathcal{C})$$

- Stemming from the dynamical system theory, a fair measure of the ***dynamical*** complexity of a system is given by Kolmogorov-Sinai entropy $h_{\rm KS}$ (in the long time limit):

$$h_{\rm KS}=\lim_{t\to\infty}-\frac{1}{t}\sum_{\text{histories}}\text{Prob}\{\text{history}\}\ln\text{Prob}\{\text{history}\}$$

KS entropy and Markov dynamics

Following Gaspard, one can discretize a continuous-time process in small time slices and compute the Kolmogorov-Sinai entropy by paralleling the usual dynamical system theory. However, the corresponding $h_{\rm KS}^{\rm discrete}(\tau)$ is not well defined when sending the time step τ to zero:

$$h_{\rm KS}^{\rm discrete}(\tau)=\ln\Big(\frac{1}{\tau}\Big)\sum_{\mathcal{C},\mathcal{C}'}P_{\rm st}(\mathcal{C})\,W(\mathcal{C}\to\mathcal{C}')-\sum_{\mathcal{C},\mathcal{C}'}P_{\rm st}(\mathcal{C})\,W(\mathcal{C}\to\mathcal{C}')\ln W(\mathcal{C}\to\mathcal{C}')$$

To give $h_{\rm KS}$ a unit-independent definition, the quantity Prob{history} has to be dimensionless. One possibility is to consider the probability of an history $\mathcal{C}_0\to\ldots\to\mathcal{C}_K$ of different successive configurations, which writes

$$\text{Prob}(\mathcal{C}_0\to\ldots\to\mathcal{C}_K)=\frac{W(\mathcal{C}_0\to\mathcal{C}_1)}{r(\mathcal{C}_0)}\cdots\frac{W(\mathcal{C}_{K-1}\to\mathcal{C}_K)}{r(\mathcal{C}_{K-1})}$$

where $r(\mathcal{C})=\sum_{\mathcal{C}'}W(\mathcal{C}\to\mathcal{C}')$ represents the escape rate from configuration $\mathcal{C}$. Between time 0 and t, the number K of jumps among configurations is not fixed. One can thus define

$$h_{\rm KS}\equiv-\frac{1}{t}\Big\langle\text{Prob}(\mathcal{C}_0\to\ldots\to\mathcal{C}_K)\ln\text{Prob}(\mathcal{C}_0\to\ldots\to\mathcal{C}_K)\Big\rangle_{\rm jumps}$$

where $\langle\cdot\rangle_{\rm jumps}$ represents the averaging over the number of events and over the durations between two successive events.

An example: the ferromagnet

As an illustration, we consider the infinite range Ising ferromagnet, composed of N spins thermostated at inverse temperature β.

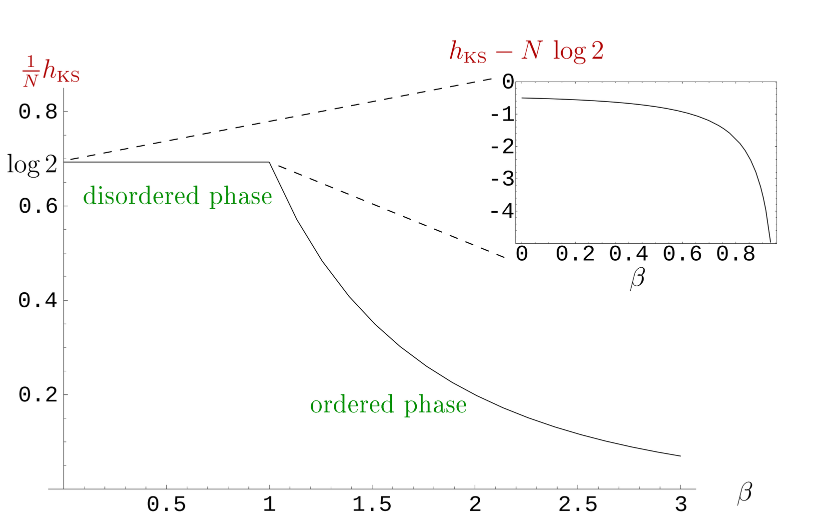

Fig. 1: KS entropy of the infinite range Ising ferromagnet (with $N\gg1$ spins), as a function of the inverse temperature β.

As indicated in **fig. 1**, the existence of a (static) phase transition between an ordered phase ($\beta>1$) and a disordered phase ($\beta<1$) is well reflected in the behaviour of the KS entropy, which is constant (at order N) in the disordered phase. Moreover, as expected from common sense, we observe that the ordered phase is dynamically less chaotic than the disordered one.

Time-reversed KS entropy and entropy flux

Reversing the arrow of time may change the way in which systems get dynamically more complex. To characterize the chaoticity of a system observed backwards in time, one is interested in looking at the ***time-reversed*** KS entropy $h_{\rm KS}^R$. Both $h_{\rm KS}$ and $h_{\rm KS}^R$ can be extracted from the long-time behaviour of the history-dependent observables $Q_\pm$:

$$Q_\pm(t)=\sum_{k=0}^{K-1}\ln\frac{W(\mathcal{C}_k\rightleftarrows\mathcal{C}_{k+1})}{r(\mathcal{C}_k)}$$

as we have (with $t\to\infty$)

$$h_{\rm KS}=-\frac{\langle Q_+\rangle}{t}\qquad\text{and}\qquad h_{\rm KS}^R=-\frac{\langle Q_-\rangle}{t}$$

where $\langle\cdot\rangle$ stands for the average over the histories spanning from 0 to t. As pointed out by Gaspard in the context of discrete-time Markov processes, the difference between $h_{\rm KS}^R$ and $h_{\rm KS}$ is equal to the entropy current flowing through the system. In our formalism, the same equality holds at the level of the fluctuating observables:

$$Q_+(t)-Q_-(t)=\sum_{k=0}^{K-1}\ln\frac{W(\mathcal{C}_k\to\mathcal{C}_{k+1})}{W(\mathcal{C}_{k+1}\to\mathcal{C}_k)}=Q_S(t)$$

The quantity $Q_S(t)$ is the integrated Lebowitz-Spohn-Gaspard-Maes entropy current. It represents the entropy variation due to the exchange of particles and/or energy between the system and the environment.

$$S[P(t)]-S[P(0)]=\underbrace{\Sigma(t)}_{\substack{\text{positive}\\\text{production}}}-\underbrace{\langle Q_S(t)\rangle}_{\substack{\text{exchange}\\\text{entropy}}}$$

State-dependent KS entropy

From the master equation, $h_{\rm KS}$ appears as the mean value, *in the stationary state*, of an instantaneous observable $J_+(\mathcal{C})$

$$h_{\rm KS}=-\langle J_+\rangle_{\rm st}\qquad\text{with}\qquad J_+(\mathcal{C})=\sum_{\mathcal{C}'}W(\mathcal{C}\to\mathcal{C}')\ln\frac{W(\mathcal{C}\to\mathcal{C}')}{r(\mathcal{C})}$$

where $\langle X\rangle_{\rm st}=\sum_{\mathcal{C}}XP_{\rm st}(\mathcal{C})$ and $P_{\rm st}$ is the stationary state. This expression strongly suggests to define a state-dependent KS entropy

$$h_{\rm KS}[P]=-\langle J_+\rangle_P=-\sum_{\mathcal{C}}P(\mathcal{C})J_+(\mathcal{C})$$

where $P(\mathcal{C})$ now represents any probability distribution.

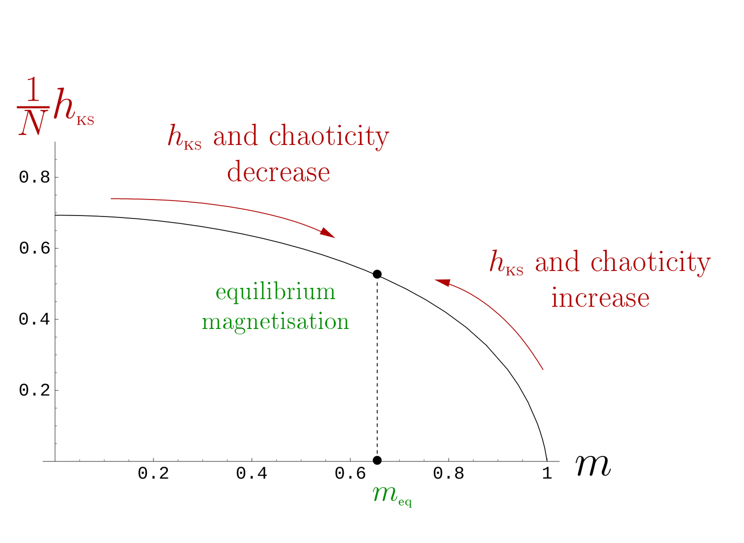

Fig. 2: $h_{\rm KS}[P]$ for the infinite range Ising, in a state of mean magnetisation m, as a function of m, at $\beta=1.2$. The equilibrium magnetisation $m_{\rm eq}\simeq0.66$ is solution of the mean-field equation $m_{\rm eq}=\tanh\beta m_{\rm eq}$.

For the Ising ferromagnet (**fig. 2**), $h_{\rm KS}[P]$ only depends on the mean magnetisation m in the state $P(\mathcal{C})$. This example illustrates how the KS entropy evolves when the systems starts from a completely disordered state ($m=0$) [or in an ordered state ($m=1$)] before aquiring, in the long-time limit, its equilibrium magnetisation $m_{\rm eq}$.

A "dynamical free energy": the topological pressure $\psi(s)$

- In equilibrium, the Gibbs-Boltzman partition function reads

$$Z_{\rm eq}(\beta)=\sum_{\mathcal{C}}\mathrm{e}^{-\beta\mathcal{H}[\mathcal{C}]}=\mathrm{e}^{N\,f(\beta)}\qquad(\text{large }N)$$

Configurational phase transitions give birth to non-analyticities in the mean free-energy $f(\beta)$.

- Stemming from the dynamical system theory, the dynamical partition function reads

$$Z_{\rm dyn}(s,t)=\sum_{\substack{\text{histories}\\\text{from 0 to }t}}\text{Prob}\{\text{history}\}^{1-s}=\mathrm{e}^{t\psi(s)}\qquad(\text{large }t)$$

The parameter s enables to probe low or high probability ***histories***. The topological pressure $\psi(s)$ plays the rôle of a dynamical free energy.

In the context of continuous-time Markov processes, the dynamical partition function $Z_{\rm dyn}(s,t)$ appears as the generating function of the cumulants of the observable $Q_+(t)$:

$$Z_{\rm dyn}(s,t)=\Big\langle e^{-sQ_+(t)}\Big\rangle$$

The topological pressure $\psi(s)$ appears as the largest eigenvalue of the operator:

$$\mathbb{W}_+(\mathcal{C},\mathcal{C}')=W(\mathcal{C}'\to\mathcal{C})^{1-s}r(\mathcal{C}')^s-r(\mathcal{C})\delta_{\mathcal{C},\mathcal{C}'}$$

Topological pressure of the ferromagnet

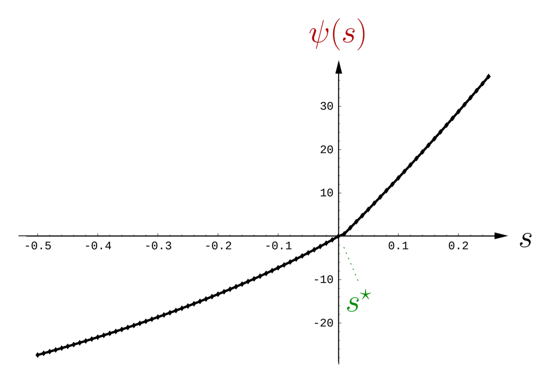

Fig. 3: the topological pressure $\psi(s)$ (or "dynamical free energy") of the Ising model in the ordered phase ($\beta=1.4$).

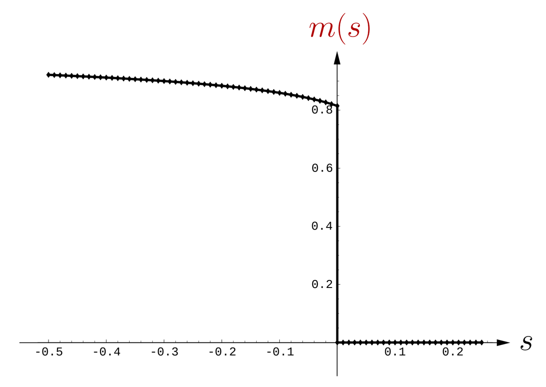

Fig. 4: the dynamical order parameter $m(s)$ corresponding to the topological pressure $\psi(s)$ of the Ising ferromagnet.

- In the low temperature phase, the topological pressure $\psi(s)$ (**fig. 3**) is not analytic at $s^\star=0$. This **dynamical phase transition** comes from the fact that
  - histories more chaotic than the stationary ones are similar to disordered-phase histories ($m(s)=0$).
  - histories less chaotic than the stationary ones are similar to ordered-phase histories ($0<m(s)<m_{\rm eq}$).

A system with an absorbing state: the contact process

Consider an (infinite range) contact process: $N\gg1$ sites are either free or occupied by a particle, and each site is subject to the following Markov transition rules:

- each occupied site becomes empty with rate 1
- each empty site becomes occupied with rate $\lambda n/N$, where n is the total number of particles.

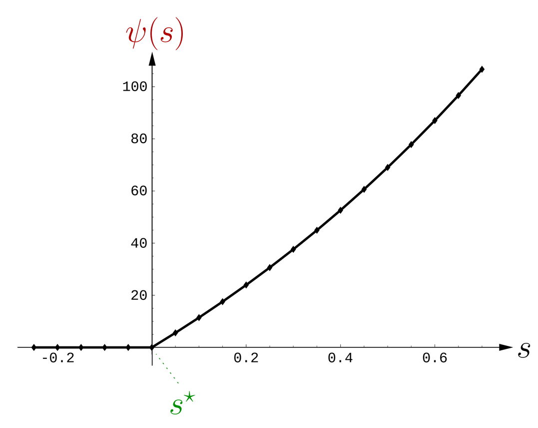

Fig. 5: the topological pressure $\psi(s)$ (or "dynamical free energy") of the contact process ($\lambda=5$).

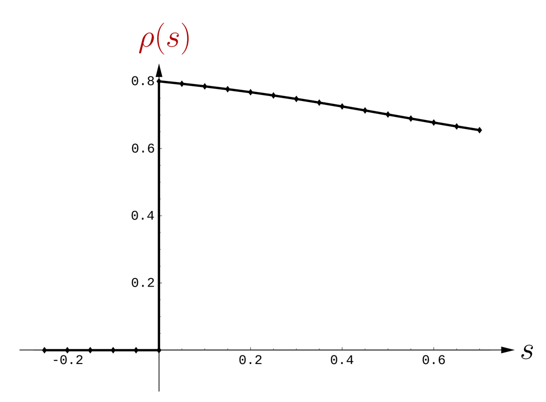

Fig. 6: the dynamical order parameter $\rho(s)$ corresponding to the topological pressure $\psi(s)$ of the contact process.

Again, the topological pressure $\psi(s)$ (**fig. 5**) is not analytic at $s^\star=0$, which reflects the presence of two dynamical phases in the thermodynamic limit:

- an active phase of non-zero density $\rho=1-\frac{1}{\lambda}$
- an absorbing state in which the system does not evolve anymore

The density $\rho(s)$ plays here the rôle of a *dynamical order parameter* with respect to the transition between the active and inactive phase.

Summary and perspectives

- Meaningful extension of dynamical system theory to continuous-time stochastic processes.
- A tool to probe statistics of histories: thermodynamics in phase space.
- Extension to systems whose dynamics is governed by rare events.

References

[1] Vivien Lecomte, Cécile Appert-Rolland and Frédéric van Wijland, *Chaotic Properties of Systems with Markov Dynamics* Phys. Rev. Lett. 95, 010601 (2005) `[cond-mat/0505483]`