Large deviations in the Symmetric Exclusion Process

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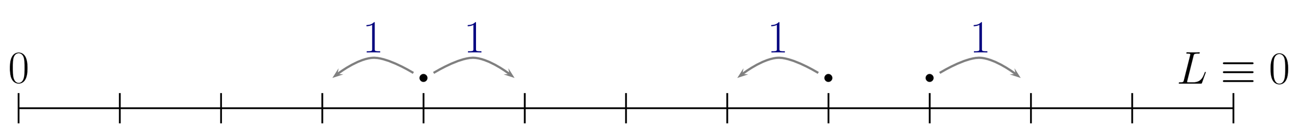
Abstract

We analyze the large deviation properties in the simple symmetric exclusion process (SSEP) with periodic boundary conditions. Our interest goes to the total current and to the number of configuration changes the system has undergone over an asymptotically large time window. Exact results are provided for the one-dimensional case.

We show how universal features, reaching beyond the particular case of the SSEP, are seen to emerge using a fluctuating hydrodynamics approach [4].

Model and Quantities of Interest



- Model: Simple Symmetric Exclusion Process (SSEP)
  - ⋆ 1D lattice of L sites with periodic boundary conditions
  - ⋆ Each site i is empty ($n_i = 0$) or occupied ($n_i = 1$)
  - ⋆ Dynamics: particles hop to their right [left] neighbor (if unoccupied) with rate 1
  - ⋆ Fixed total number N of particles
- (History-dependent) time-extensive observables

K = | number of particle jumps between 0 and t (Activity)

Q = | total current of particles between 0 and t (Current)

Large Deviation Functions

- Cumulant generating functions $\psi_K(s) = \lim_{t\to\infty} \frac{1}{t}\langle e^{-sK}\rangle \quad \psi_Q(s) = \lim_{t\to\infty} \frac{1}{t}\langle e^{-sQ}\rangle$
- Observables in the s-ensemble (s conjugated to K or Q)

$$\langle \mathcal{O}\rangle_s = \frac{\langle \mathcal{O}\, e^{-sK}\rangle}{\langle e^{-sK}\rangle} \qquad \langle \mathcal{O}\rangle_s = \frac{\langle \mathcal{O}\, e^{-sQ}\rangle}{\langle e^{-sQ}\rangle}$$

- Evolution equation (for K)

$$\partial_t P(\mathcal{C},K,t) = \sum_{\mathcal{C}'} W(\mathcal{C}'\to\mathcal{C})P\big(\mathcal{C}',K-1,t\big) \; - r(\mathcal{C})P(\mathcal{C},K,t)$$

($\mathcal{C}$ = configuration of the system)

- Laplace transform $\hat{P}(\mathcal{C},s,t) = \sum_K e^{-s}P(\mathcal{C},K,t)$ evolves with $\partial_t P_K = \mathbb{W}_K P_K$

$$\big(\mathbb{W}_K\big)_{\mathcal{C}',\mathcal{C}} = e^{-s}\,W(\mathcal{C}\to\mathcal{C}') - r(\mathcal{C})\delta_{\mathcal{C}\mathcal{C}'} \qquad \text{(evolution operator)}$$

with $r(\mathcal{C}) = \sum_{\mathcal{C}'} W(\mathcal{C}\to\mathcal{C}')$= escape rate from $\mathcal{C}$

- Maximum eigenvalues: $\psi_K(s) = \max \mathrm{Sp}\, \mathbb{W}_K\,, \quad \psi_Q(s) = \max \mathrm{Sp}\, \mathbb{W}_Q$

Series Expansion

$$\psi_K(s)\langle \mathcal{O}(\mathcal{C})\rangle_s = e^{-s}\Big\langle \sum_{\mathcal{C}'} W(\mathcal{C}\to\mathcal{C}')\mathcal{O}(\mathcal{C}')\Big\rangle_s - \langle \mathcal{O}(\mathcal{C})r(\mathcal{C})\rangle_s$$

- Expansion in powers of s, quantum mechanics like perturbation theory (as in [1]):

$$\frac{\langle K\rangle}{t} = \frac{L^2\sigma}{L-1} \sim L\sigma, \qquad \frac{\langle K^2\rangle_c}{t} = \frac{L^2\sigma(\frac{1}{2}L^2\sigma + 2L - 2)}{3(L-1)^2} \sim \frac{1}{6}\sigma^2 L^2,$$
$$\frac{\langle Q^2\rangle_c}{t} = \frac{L^2\sigma}{L-1} \sim L\sigma, \qquad \frac{\langle Q^4\rangle_c}{t} = \frac{1}{2}\frac{L^4\sigma^2}{(L-1)^2} \sim \frac{1}{2}\sigma^2 L^2,$$

- Large size limit: $N, L \to \infty$, $\rho = N/L$ fixed, $\sigma(\rho) = 2\rho(1-\rho)$
- Surprisingly, $\langle K\rangle \sim L$, $\langle Q^2\rangle_c \sim L$ **BUT** $\langle K^2\rangle_c \sim L^2$ $\langle Q^4\rangle_c \sim L^2$
- Hint for a **common universal scaling function** $\mathcal{F}(x) = \frac{1}{3}x^2 + \frac{1}{45}x^3 + \frac{1}{378}x^4 + \dots$

$$\lim_{L\to\infty} L^2\big[\psi_K(s) + s\tfrac{1}{t}\langle K\rangle\big] \;=\; \mathcal{F}\big(u = \tfrac{1}{2}\sigma L^2 s\big)$$
$$\lim_{L\to\infty} L^2\big[\psi_Q(s) - \tfrac{1}{2}s^2\tfrac{1}{t}\langle Q^2\rangle_c\big] \;=\; \mathcal{F}\big(v = -\tfrac{1}{4}\sigma L^2 s^2\big)$$

Asymptotics: Bethe Ansatz

- Formulation in terms of a XYZ Hamiltonian

$$-\mathbb{W}_K = \frac{L}{2} - \frac{1}{2}\sum_{i=1}^{L}\big[e^{-s}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + \sigma_i^z\sigma_{i+1}^z\big]$$
$$-\mathbb{W}_Q = \frac{L}{2} - \frac{1}{2}\sum_{i=1}^{L}\big[\mathrm{ch}\, s\,(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) + \sigma_i^z\sigma_{i+1}^z - i\,\mathrm{sh}\, s\,(\sigma_i^x\sigma_{i+1}^y - \sigma_i^y\sigma_{i+1}^x)\big]$$

- Bethe diagonalization (x_i = position of the i^{th} particle, $\mathcal{P}$ = permutation of $[1,N]$):

$$P(\{n_j\},s) = \sum_{\mathcal{P}}\mathcal{A}(\mathcal{P})\prod_{i=1}^{N}\big[\zeta_{p(i)}\big]^{x_i} \quad \text{with} \quad \zeta_i^L = \prod_{\substack{j=1\\ j\neq i}}^{N}\left[-\frac{1-2e^s\zeta_i + \zeta_i\zeta_j}{1-2e^s\zeta_j+\zeta_i\zeta_j}\right] \quad \text{(for } K\text{)}$$

- Wave-numbers [2] $\zeta_j = e^{ik_j\delta}$, $\cos\delta = e^s$, perturbation theory in s
- Asymptotics: $\frac{1}{L}\psi_K(s) \;=\; -2\rho(1-\rho)s \;+\; \frac{2^{7/2}}{3\pi}[\rho(1-\rho)]^{3/2}|s|^{3/2} + \mathcal{O}(s^2)$
- For Q [2, 3]: $\frac{1}{L}\psi_Q(s) \;=\; \frac{1}{2}\sigma s^2 \;+\; \frac{(2\pi)^{2/3}}{20}\sigma^{4/3}|s|^{8/3} + \mathcal{O}(s^3) \qquad (L^{-3/4} \ll |s| \ll 1)$

Fluctuating Hydrodynamics

- Coarse-grained density $\rho(x,\tau) = n_{j=xL}$ in a continuum space ($0 \le x \le 1$).
- Density fluctuation $\phi(x,\tau) = \rho(x,\tau) - \rho$ evolve with nonlinear Langevin equation

$$\partial_\tau\phi(x,\tau) = -\partial_x j(x,\tau), \quad j(x,t) = -\partial_x\phi + \xi(x,\tau)$$
$$\langle \xi(x,\tau)\xi(x',\tau')\rangle = L^{-1}\sigma(\rho(x,\tau))\delta(x-x')\delta(\tau-\tau')$$

- Macroscopic activity and current

$$K = 2L^3\int_0^T d\tau\int_0^1 dx\; \rho(x,\tau)(1-\rho(x,\tau))$$
$$Q = L\int_0^T d\tau\int_0^1 dx\; j(x,\tau) = L\int_0^T d\tau\int_0^1 dx\; \xi(x,\tau)$$

Universal Scaling Functions

- K quadratic in ξ: $\langle e^{-sK}\rangle \leftrightarrow$ **Gaussian integral** (assuming small fluct.)
- Confirmation: **universal scaling function**

$$\mathcal{F}\big(u = \tfrac{1}{2}\sigma L^2 s\big) = \lim_{L\to\infty} L^2\big[\psi_K(s) + s\tfrac{1}{t}\langle K\rangle\big]$$
$$\mathcal{F}(u) = 4\pi^2\sum_{n\ge 1}\Big\{n^2 - \sqrt{n^2\big(n^2 - \tfrac{2}{\pi^2}u\big)} - \tfrac{1}{\pi^2}u\Big\} = \sum_{k\ge 2}\frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

- Analytic continuation

$$\mathcal{F}(u) = \begin{cases} 2\displaystyle\int_0^u dx\left[1 - \frac{1}{\pi}\int_0^1 dy\,\frac{\sqrt{2x}\,\cot\sqrt{2xy}}{\sqrt{1-y}}\right] & \text{for } 0 < u < \frac{\pi^2}{2} \\ 2\displaystyle\int_0^u dx\left[1 - \frac{1}{\pi}\int_0^1 dy\,\frac{\sqrt{-2x}\,\coth\sqrt{-2xy}}{\sqrt{1-y}}\right] & \text{for } u < 0 \end{cases}$$

- For K: transition at $u = \frac{\pi^2}{2}$ (inactive configurations $\leftrightarrow$ clustering)
- For Q: $\mathcal{F}(v)$ valid only for $|s| \ll L^{-3/4}$ (crossover from Edwards-Wilkinson to KPZ)

References

[1] B. Derrida, B. Douçot, P.E. Roche, J. Stat. Phys. **115**, 717-748 (2004), *Current fluctuations in the one dimensional symmetric exclusion process with open boundaries*

[2] L.H. Gwa and H. Spohn, Phys. Rev. Lett. **68** 725 (1992), *Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian*; D. Kim, Phys. Rev. E **52** 3512 (1995), *Bethe ansatz solution for crossover scaling functions of the asymmetric XXZ chain and the KPZ-type growth model*

[3] J.L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999), *A Gallavotti-Cohen Type Symmetry in the Large Deviation Functional for Stochastic Dynamics*

[4] C. Appert-Rolland, B. Derrida, V. Lecomte and F. van Wijland, *Large deviations in the symmetric exclusion process*, in preparation