Supersymmetries in nonequilibrium Langevin dynamics

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Stochastic phenomena are often described by Langevin equations, which serve as a mesoscopic model for microscopic dynamics. It has been known since the work of Parisi and Sourlas that *reversible* (or equilibrium) dynamics present supersymmetries (SUSYs). These are revealed when the path-integral action is written as a function not only of the physical fields, but also of Grassmann fields representing a Jacobian arising from the noise distribution. SUSYs leave the action invariant upon a transformation of the fields that mixes the physical and the Grassmann ones. We show that contrary to common belief, it is possible to extend the known reversible construction to the case of arbitrary *irreversible* dynamics, for overdamped Langevin equations with additive white noise—provided their steady state is known. The construction is based on the fact that the Grassmann representation of the functional determinant is not unique, and can be chosen so as to present a generalization of the Parisi-Sourlas SUSY. We show how such SUSYs are related to time-reversal symmetries and allow one to derive modified fluctuation-dissipation relations valid in nonequilibrium. We give as a concrete example the results for the Kardar-Parisi-Zhang equation.

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I. INTRODUCTION

The dynamics of a large number of elementary constituents can often be described by mesoscopic stochastic equations of motion, where the effects of interactions at small scales are accounted for by friction and noise. Such an effective Langevin [1] description applies to various examples, ranging from particles in a fluid to chemical or economical processes [2,3] or cosmological inflation [4,5].

Field theory then allows one to write the probability of trajectories followed by the system using a path-integral representation that encompasses both classical and quantum problems [6]. The weight of a trajectory takes the form of the exponential of (minus) an action. It is convenient to make the action depend not only on the physical fields, but also on noncommuting auxiliary ones—known as Grassmann fields—representing a Jacobian arising from the noise distribution. This action possesses a generic "supersymmetry" (SUSY), known as the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [7–10]. It encodes the conservation of probability. Also, when the dynamics is *reversible* (i.e., forces derive from a potential), a second SUSY was uncovered by Parisi-Sourlas [11] and by Feigel'man-Tsvelik [12] (after a similar SUSY was found for the partition function of equilibrium problems [13]).

Such SUSYs, which mix physical and Grassmann fields, look surprising in a statistical mechanical context; yet, as other symmetries in physics, they turn out to be a powerful tool to study a variety of problems. These range from the dynamics of spin glasses [14,15], disordered spin models [16], or heteropolymers [17], to finite-size effects in critical dynamics [18], localization [19], renormalization of the random-field Ising model [20–22], symmetries of Hamiltonian dynamics [23,24], and metastability in overdamped [25] and inertial [26] Langevin dynamics, with Witten's SUSY version of Morse theory [27]. SUSYs have methodological implications for renormalization [28] and the derivation of variational principles [29] or of the Parisi-Wu stochastic quantization [30–32]. The Parisi-Sourlas SUSY implies Ward identities yielding the equilibrium fluctuation-dissipation relation (FDR) [33,34]. When the dynamics is irreversible, the BRST symmetry remains valid, but the Parisi-Sourlas one is broken e.g., by a driving field [35,36] or a colored noise [37]. It has been argued indeed that microreversibility is at the origin of SUSY [34].

In this paper, we prove the contrary, by extending the previously known results to the case of arbitrary nonequilibrium Langevin dynamics (in the overdamped limit and for additive Gaussian white noise). We assume that the stationary distribution exists and our construction depends explicitly on it. The key observation is that there are several inequivalent ways to represent the same Jacobian through Grassmann fields, and we identify one that presents an extended SUSY generalizing the Parisi-Sourlas one. We show that the associated Ward identities yield modified FDRs, recovering some known cases [38–40]. Then, we explain how this SUSY is directly related to a time-reversal symmetry between the original Langevin dynamics and its "adjoint." We identify the mathematical structure at the origin of the extended SUSY. The construction can be carried out both in the Martin-Siggia-Rose-Janssen-de Dominicis (MSRJD) framework [41–45] and in the Onsager-

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Machlup one [46,47], where it takes a particularly simple form. Finally, we discuss the cases of spatially correlated noise, continuum space, and the example of the Kardar-Parisi-Zhang (KPZ) equation [48].

II. BRST SUSY

Consider a set of scalar fields $h_i(t)$ evolving in time according to a Langevin equation,

$$\partial_t h_i = f_i[h] + \eta_i, \tag{1}$$

where $f_i[h]$ is a deterministic force function of the fields $h = (h_i)$ at time t, and $\eta_i(t)$ is a centered Gaussian white noise with $\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t'-t)$ (the generalization to anisotropic correlated noise is detailed below). For instance, $h_i(t)$ represents the spatial coordinate of a particle tagged by a discrete index i or the value of the height of an interface on a lattice site i (as in the KPZ equation). Equation (1) is equivalent to a Fokker-Planck evolution $\partial_t P[h,t] = \mathbb{W}P[h,t]$ for the distribution $P[h,t]$ of h, with

$$\mathbb{W}\cdot = -\partial_i[f_i[h]\cdot - T\partial_i\cdot]. \tag{2}$$

We denote $\partial_i \equiv \frac{\partial}{\partial h_i}$ and use implicit summation over repeated indices (including in squares such as X_i^2). We assume that the dynamics possesses a stationary distribution $P_{\rm st}[h]$ such that $\mathbb{W}P_{\rm st} = 0$, and define a functional $\mathcal{H}[h]$ by $P_{\rm st}[h] \propto \mathrm{e}^{-\frac{1}{T}\mathcal{H}[h]}$. This is the so-called quasipotential, which exists under generic conditions [49]. Then, following Graham [50] and Eyink *et al.* [51], we decompose the total force as the sum of a conservative force $-\partial_i\mathcal{H}[h]$ and a driving force $g_i[h]$ as

$$f_i[h] = -\partial_i\mathcal{H}[h] + g_i[h]. \tag{3}$$

The case of reversible dynamics is recovered for $g_i[h] \equiv 0$. This decomposition is generic when the quasipotential exists. From (2), the stationary condition $\mathbb{W}P_{\rm st} = 0$ is equivalent to an identity that will be used thoroughly,

$$\partial_i g_i[h] = \frac{1}{T} g_i[h]\,\partial_i\mathcal{H}[h]. \tag{4}$$

We consider the distribution of fields on a finite time window $[0, t_{\rm f}]$ and denote $\int_t = \int_0^{t_{\rm f}} dt$ (but this time window can also be $\mathbb{R}$). The path-integral representation [6] of the trajectory probability follows from a mere change of variable from the Gaussian noise distribution, $\mathrm{Prob}[\eta] \propto e^{-\int_t \eta_i^2/(4T)}$, to that of the field h, seen from the Langevin equation (1) as a functional of the noise,

$$P[h] = \left|\frac{\delta\eta}{\delta h}\right| e^{-\frac{1}{4T}\int_t \eta_i[h]^2}, \qquad \eta_i[h] \equiv \partial_t h_i - f_i[h]. \tag{5}$$

Here, $\eta_i[h]$ is the expression of the noise as a function of h in the Langevin equation (1), and $|\frac{\delta\eta}{\delta h}| = |\det\frac{\delta\eta_i[h(t)]}{\delta h_j(t')}|$ is the functional Jacobian of the change of variables from η to h. We emphasize that even if the Langevin equation (1) is additive and does not depend on its time discretization, the expressions of the Jacobian and of the path-integral action do depend on the discretization chosen to write them [52–54]. We adopt the Stratonovich convention, which allows one to use the rules of calculus in the path integral [55] and to reverse time without changing the discretization [56,57]. Following Janssen [41], one then linearizes the square in the exponent of (5) using a "response field" $\hat{h}_i(t)$ to obtain the MSRJD action. Introducing anticommuting Grassmann fields $\overline{\Psi}_i(t)$ and $\Psi_i(t)$ [58] to represent $|\frac{\delta\eta}{\delta h}|$, we get

$$P[h] = \int \mathcal{D}\hat{h}\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\, e^{-S_{\rm SUSY}}, \tag{6}$$

$$S_{\rm SUSY} = \int_t \big\{\hat{h}_i\eta_i[h] - T\hat{h}_i^2 - \overline{\Psi}_i\eta_i'[h]\Psi\big\}. \tag{7}$$

The response field $\hat{h}_i$ is integrated on the imaginary axis, and $\eta_i'[h]$ is the Fréchet derivative of $\eta_i[h]$ which is a linear operator acting on the vector Ψ as $\eta_i'[h]\Psi = \partial_j\eta_i[h]\Psi_j$ [59]. The BRST SUSY, which originates in the conservation of probability, is a Grassmann symmetry: it depends on a Grassmann parameter ε that allows one to mix the anticommuting Grassmann and the commuting physical fields as $h \mapsto h + \delta h$, $\hat{h} \mapsto h + \delta\hat{h}$, etc., with

$$\text{BRST:}\ \ \delta h_i = \varepsilon\Psi_i, \quad \delta\hat{h}_i = 0, \quad \delta\overline{\Psi}_i = \varepsilon\hat{h}_i, \quad \delta\Psi_i = 0. \tag{8}$$

$S_{\rm SUSY}$ is invariant under (8) since $\delta(\eta_i[h]) = \eta_i'[h]\delta h = \varepsilon\eta_i'[h]\Psi$. (We denote $\delta(X) = X[h+\delta h, \dots] - X[h, \dots]$).

III. EXTENDED PARISI-SOURLAS SUSY

When forces derive from a potential ($g_i[h] \equiv 0$), another SUSY was found by Parisi-Sourlas [11] and by Feigel'man-Tsvelik [12], in relation with the former work of Nicolai [60–62] (see [63]). It yields the equilibrium FDR [33,34] (as discussed below). We now extend these results to the generic Langevin dynamics (1). The key observation is that one can identify a Grassmann action, different from (7), but that still fully represents the Langevin equation (1) and possesses a SUSY,

$$S_{\rm SUSY}^\dagger = \int_t \bigg\{\hat{h}_i\eta_i - T\hat{h}_i^2 + \frac{1}{T}g_i\partial_i\mathcal{H} - \overline{\Psi}_i\tilde{\eta}_i'^\dagger\Psi\bigg\}, \tag{9}$$

$$\tilde{\eta}_i[h] = \partial_t h_i + \partial_i\mathcal{H}[h] + g_i[h] \tag{10}$$

(for compaction, we drop some dependencies in h). For an operator A, we set $(A^\dagger)_{ij} = A_{ji}$ [64]. The Grassmann part of $S_{\rm SUSY}^\dagger$ involves $\tilde{\eta}_i[h]$, whose signification as the noise of an "adjoint" dynamics becomes clear below when relating SUSYs to time reversal.

For a reversible dynamics ($g_i[h] \equiv 0$), one sees that $S_{\rm SUSY} = S_{\rm SUSY}^\dagger$: the actions (7) and (9) are identical. For an arbitrary irreversible dynamics ($g_i[h] \neq 0$), one has $S_{\rm SUSY} \neq S_{\rm SUSY}^\dagger$, and yet, as we now show in detail, *the actions* (7) *and* (9) *represent the same Langevin equation* (1) [*and thus the same Fokker-Planck operator* (2)]. This is due to the fact that when integrating over $\overline{\Psi}, \Psi$, the extra term $\frac{1}{T}g_i\partial_i\mathcal{H}$ in (9) ensures that the Jacobian $|\frac{\delta\eta}{\delta h}|$ is correctly represented. To show this, we first recall that in Stratonovich discretization [28,42,65–71],

$$\left|\frac{\delta\eta[h]}{\delta h}\right| = \exp\bigg\{-\frac{1}{2}\int_t \mathrm{tr}\, f_i'[h]\bigg\}, \tag{11}$$

which can be obtained by time discretization [72], and where the trace is $\mathrm{tr}\, f_i'[h] = \partial_i f_i[h]$. The time discretization of $\overline{\Psi}, \Psi$ in (7) is crucial to correctly represent the Jacobian (11) [73]. Integrating over the fields $\overline{\Psi}, \Psi$ in $e^{-S_{\rm SUSY}^\dagger}$ yields $|\frac{\delta\tilde{\eta}}{\delta h}| =$

$e^{\frac{1}{2}\int_t \mathrm{tr}(\partial_i\mathcal{H}+g_i)'[h]}$; see Eq. (11). Thus, the contribution of the last two terms in (9) is $e^{-\int_t\{\frac{1}{T}g_i\partial_i\mathcal{H}-\frac{1}{2}\mathrm{tr}(\partial_i\mathcal{H}+g_i)'\}}$, but this expression is in fact equal to the Jacobian (11) because the stationary condition (4) implies $\frac{1}{T}g_i\partial_i\mathcal{H} = \mathrm{tr}\, g_i'$. We thus have shown that $\int \mathcal{D}\hat{h}\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\, e^{-S_{\mathrm{SUSY}}} = \int \mathcal{D}\hat{h}\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\, e^{-S^\dagger_{\mathrm{SUSY}}}$.

Hence, despite being different in general, the actions S_{SUSY} and $S^\dagger_{\mathrm{SUSY}}$ both correctly represent the trajectory probability of the Langevin equation (1) (and we denote by $\langle\cdot\rangle$ and $\langle\cdot\rangle^\dagger$ the corresponding averages). Physically, this means that observables depending only on h and $\hat{h}$ have the same average: $\langle\mathcal{O}[h,\hat{h}]\rangle = \langle\mathcal{O}[h,\hat{h}]\rangle^\dagger$. This is, of course, not the case if $\mathcal{O}$ depends on Ψ or $\overline{\Psi}$. This freedom of representation originates in the fact that the Jacobian depends only on the diagonal components of the operator $\eta_i'[h]$ through the trace $\mathrm{tr}\, f_i'[h] = \partial_i f_i[h]$, and not on all its components, $(\eta_i'[h])_j = \delta_{ij}\partial_t - \partial_j f_i[h]$ [74].

Then, one checks by direct computation that

$$\mathrm{PS}_1\colon \begin{cases} \delta h_i = \varepsilon T\,\overline{\Psi}_i, & \delta\hat{h}_i = \varepsilon(\delta_{ij}\partial_t - \partial_j g_i[h])\overline{\Psi}_j, \\ \delta\Psi_i = \varepsilon(\partial_t h_i - g_i[h] - T\hat{h}_i), & \delta\overline{\Psi}_i = 0 \end{cases} \qquad (12)$$

leaves $S^\dagger_{\mathrm{SUSY}}$ invariant, up to time-boundary terms. This SUSY generalizes the Parisi-Sourlas one to arbitrary irreversible dynamics (1) since, for reversible dynamics ($g_i[h]\equiv 0$), we have $S^\dagger_{\mathrm{SUSY}} = S_{\mathrm{SUSY}}$, and (12) yields the known SUSY [11]. An important difference with the reversible case $g_i[h]\equiv 0$ is that this transformation is now nonlinear in general because of the terms $\propto g_i[h]$ in (12). We also uncover a dual SUSY,

$$\mathrm{PS}_2 : \begin{cases} \delta h_i = \varepsilon T\,\overline{\Psi}_i, & \delta\hat{h}_i = \varepsilon\partial_{ij}\mathcal{H}[h]\overline{\Psi}_j, \\ \delta\Psi_i = -\varepsilon(\partial_i\mathcal{H}[h] - T\hat{h}_i), & \delta\overline{\Psi}_i = 0, \end{cases} \qquad (13)$$

which seems to have been unnoticed even for $g_i[h]\equiv 0$ (perhaps because it is nonlinear, even in this case).

We emphasize that this construction can also be formulated using the superfield, with explicit expressions for the generators of $\mathrm{PS}_{1,2}$ [75]. One can also transpose it to the Onsager-Machlup formalism straightforwardly: indeed, the passage from the MSRJD to the Onsager-Machlup action is done by integrating over the response field, which amounts to replacing $\hat{h}$ by its optimal value, $\hat{h}^{\mathrm{opt}} = \frac{1}{2T}\eta[h]$ [75]. The corresponding SUSY transformation is obtained likewise, as made explicit below.

The nonequilibrium SUSY that we derived is more intricate than in equilibrium since it involves two actions (S_{SUSY} invariant only under BRST and $S^\dagger_{\mathrm{SUSY}}$ only under $\mathrm{PS}_{1,2}$) and depends explicitly on the steady state. However, it allows one to derive physical consequences, as shown now.

IV. MODIFIED FDRs

Symmetries of the action imply Ward identities for correlation functions: denoting $h_1 = h_{i_1}(t_1)$ (and similarly for other indices, functions, or operators), the BRST symmetry (8) implies, in particular, $\langle h_1\overline{\Psi}_2\rangle = \langle(h_1+\delta h_1)(\overline{\Psi}_2+\delta\overline{\Psi}_2)\rangle$, and hence $\langle h_1\delta\overline{\Psi}_2\rangle + \langle\delta h_1\overline{\Psi}_2\rangle = 0$, which means

$$\langle h_1\hat{h}_2\rangle = -\langle\Psi_1\overline{\Psi}_2\rangle, \qquad (14)$$

and we find that the two-point correlator of the Grassmann fields is a response function. In particular, these correlators are 0 for $t_1 < t_2$. From the invariance of $\langle h_1\Psi_2\rangle^\dagger$ under the SUSYs $\mathrm{PS}_{1,2}$, we similarly infer

$$\langle h_1(\partial_t h_2 - g_2[h_2])\rangle = T\langle h_1\hat{h}_2\rangle - T\langle\overline{\Psi}_1\Psi_2\rangle^\dagger, \qquad (15)$$

$$\langle h_1\,\partial_2\mathcal{H}[h_2]\rangle = T\langle h_1\hat{h}_2\rangle + T\langle\overline{\Psi}_1\Psi_2\rangle^\dagger, \qquad (16)$$

where we used that for observables independent of Ψ, $\overline{\Psi}$, the actions S_{SUSY} and $S^\dagger_{\mathrm{SUSY}}$ yield the same averages. The causal structure of the Grassmann contribution to $S^\dagger_{\mathrm{SUSY}}$ shows that $\langle\overline{\Psi}_1\Psi_2\rangle^\dagger = 0$ for $t_1 > t_2$ [76] (which can also be inferred from the interpretation of $\langle\overline{\Psi}_1\Psi_2\rangle^\dagger$ as a response function in the adjoint dynamics; see below). We thus obtain two modified FDRs,

$$\langle h_1(\partial_t h_2 - g_2[h_2])\rangle = T\langle h_1\hat{h}_2\rangle \quad \text{if } t_1 > t_2, \qquad (17)$$

$$\langle h_1\,\partial_2\mathcal{H}[h_2]\rangle = T\langle h_1\hat{h}_2\rangle \quad \text{if } t_1 > t_2. \qquad (18)$$

Note that adding (15) and (16), or (17) and (18), one obtains $\langle h_1(\eta_2[h_2] - 2T\hat{h}_2)\rangle = 0$, which is always valid, as can be checked using $\delta(e^{-S_{\mathrm{SUSY}}})/\delta\hat{h}_2 = (\eta_2[h_2] - 2T\hat{h}_2)e^{-S_{\mathrm{SUSY}}}$ and a functional integration by part.

Since the right-hand side of the relation (17) is the response function $\langle h_1\hat{h}_2\rangle = \langle\delta h_1/\delta\mathfrak{f}_2\rangle_{\mathfrak{f}=0}$ to a perturbation $f\mapsto f+\mathfrak{f}$ of the total force, this relation entails a modified FDR, valid in nonequilibrium (the equilibrium one, $\langle h_1\partial_t h_2\rangle = T\langle h_1\hat{h}_2\rangle$, is recovered for $g[h]\equiv 0$ and can be derived from the Parisi-Sourlas SUSY [33,34]). A relation similar in spirit was derived in [50,51], but in a particular setting where the perturbation is acting only on the conservative part of the force, so that the left-hand side of (17) has no contribution from $g[h]$. One checks that (17) and (18) are equivalent to the Agarwal FDR [38] and its equivalent formulations (e.g., [39,40,49,77–83]). Also, Eqs. (17) and (18) and other Ward identities can be read as providing information on the quasipotential when it is not known.

V. STRUCTURE OF THE EXTENDED SUSY

Noting that $\overline{\Psi}_i(\partial_i\mathcal{H}+g_i)'[h]^\dagger\Psi = -\Psi_i(\partial_i\mathcal{H}+g_i)'[h]\overline{\Psi}$, and that integrating by parts $\int_t\overline{\Psi}_i\partial_t\Psi_i = \int_t\Psi_i\partial_t\overline{\Psi}_i$, we define a new action $\mathbb{S}^\dagger_{\mathrm{SUSY}} = S^\dagger_{\mathrm{SUSY}} - \frac{1}{T}[\mathcal{H}[h]]_0^{t_f}$ which is written as

$$\mathbb{S}^\dagger_{\mathrm{SUSY}} = \int_t\left\{\hat{h}_i\eta_i - T\hat{h}_i^2 - \frac{1}{T}(\partial_t h_i - g_i)\partial_i\mathcal{H} + \Psi_i\bar{\eta}_i'\overline{\Psi}\right\}, \qquad (19)$$

$$\bar{\eta}_i[h] = -\partial_t h_i + \partial_i\mathcal{H}[h] + g_i[h]. \qquad (20)$$

With $\frac{\eta_i+\bar{\eta}_i}{2} = \partial_i\mathcal{H}$ and $\frac{\eta_i-\bar{\eta}_i}{2} = \partial_t h_i - g_i$, one obtains

$$\mathbb{S}^\dagger_{\mathrm{SUSY}} = \int_t\left\{-\frac{1}{T}\Big(\frac{\eta_i-\bar{\eta}_i}{2} - T\hat{h}_i\Big)\Big(\frac{\eta_i+\bar{\eta}_i}{2} - T\hat{h}_i\Big) + \Psi_i\bar{\eta}_i'\overline{\Psi}\right\}. \qquad (21)$$

Such a rewriting renders manifest that $\mathbb{S}^\dagger_{\mathrm{SUSY}}$ is invariant under the SUSYs $\mathrm{PS}_{1,2}$ (without generating any boundary term). Indeed, from (12),

$$\mathrm{PS}_1 \Rightarrow \begin{cases} \delta\big(\frac{\eta_i-\bar{\eta}_i}{2} - T\hat{h}_i\big) = 0, & \delta\Psi_i = \varepsilon\delta\big(\frac{\eta_i-\bar{\eta}_i}{2} - T\hat{h}_i\big), \\ \delta\big(\frac{\eta_i+\bar{\eta}_i}{2} - T\hat{h}_i\big) = \varepsilon T\bar{\eta}_i'\overline{\Psi}, & \end{cases}$$

so that the variations of the two products in (21) cancel each other very simply. PS_2 presents a similar structure with the roles of $\eta_i + \bar{\eta}_i$ and $\eta_i - \bar{\eta}_i$ exchanged. The identified structure explains how $\partial_i \mathcal{H}$ and the "covariant derivative" $\partial_t h - g[h]$ (see [84] for KPZ) play a dual role in the SUSYs $PS_{1,2}$ and in the modified FDRs (15) and (16).

The actions $S^{\dagger}_{\rm SUSY}$ and $\mathbb{S}^{\dagger}_{\rm SUSY}$ have an equivalent physical content as they are equal up to time-boundary terms. A careful treatment of these shows that the averages in (15) and (16) on a finite time window are those sampled by the steady state $P_{\rm st}[h]$ at initial time [75].

In the Onsager-Machlup formalism, the corresponding actions are particularly simple: $S_{\rm OM} = \int_t \{\frac{\eta^2}{4T} - \overline{\Psi}\eta'\Psi\}$ and $\mathbb{S}^{\dagger}_{\rm OM} = \int_t \{\frac{\bar{\eta}^2}{4T} + \Psi\bar{\eta}'\overline{\Psi}\}$, with $S_{\rm OM}$ verifying the BRST SUSY (8), and $\mathbb{S}^{\dagger}_{\rm OM}$ being invariant by the PS SUSY $\delta h = \varepsilon T\overline{\Psi}$, $\delta\Psi = -\frac{\varepsilon}{2}\bar{\eta}$, $\delta\overline{\Psi} = 0$ corresponding to $PS_{1,2}$.

VI. TIME REVERSAL WITHOUT GRASSMANN

One can also represent $P[h]$ as a path integral on the response field only, $P[h] = \int \mathcal{D}\hat{h}\, e^{-S_{\rm MSR}}$, with the Jacobian contribution (11) included in the action,

$$S_{\rm MSR}[h,\hat{h}] = \int_t \left\{ \hat{h}_i \eta_i[h] - T\hat{h}_i^2 + \frac{1}{2}\partial_i f_i[h] \right\}. \tag{22}$$

Consider a time reversal of the field $h_i(t) = h_i^{\rm S}(t_{\rm R})$ (with $t_{\rm R} = t_{\rm f} - t$) combined with either one of these two response-field transformations (denoting $\dot{\varphi} = \partial_t \varphi$),

$$\mathrm{TR}_1:\ \hat{h}_i(t) = \hat{h}_i^{\rm S}(t_{\rm R}) - \frac{1}{T}\{\dot{h}_i^{\rm S}(t_{\rm R}) + g_i[h^{\rm S}]\}, \tag{23}$$

$$\mathrm{TR}_2:\ \hat{h}_i(t) = -\hat{h}_i^{\rm S}(t_{\rm R}) + \frac{1}{T}\partial_i \mathcal{H}[h^{\rm S}]. \tag{24}$$

The adjoint process [78,85] of (1) is the one with a force $\tilde{f}_i[h] = -\partial_i\mathcal{H}[h] - g_i[h]$ instead of $f_i[h]$. It is the process followed by time-reversed trajectories [86]. The action $\tilde{S}_{\rm MSR}$ of the adjoint process presents a mapping with $S_{\rm MSR}$

$$S_{\rm MSR}[h,\hat{h}] = \tilde{S}_{\rm MSR}[h^{\rm S},\hat{h}^{\rm S}] - \frac{1}{T}\big[\mathcal{H}[h^{\rm S}]\big]_0^{t_{\rm f}}. \tag{25}$$

To derive it, one uses the stationary condition (4) and we stress that the Jacobian and non-Jacobian contributions to the action (22) interfere. $\mathrm{TR}_{1,2}$ imply, respectively,

$$\langle h_1(\partial_t h_2 - g_2[h_2])\rangle = T\langle h_1\hat{h}_2\rangle - T\langle h_1^{\rm R}\hat{h}_2^{\rm R}\tilde{\rangle}, \tag{26}$$

$$\langle h_1\,\partial_2\mathcal{H}[h_2]\rangle = T\langle h_1\hat{h}_2\rangle + T\langle h_1^{\rm R}\hat{h}_2^{\rm R}\tilde{\rangle}, \tag{27}$$

where the superscript $^{\rm R}$ indicates that the field is time reversed and $\langle\cdot\tilde{\rangle}$ is the average for the adjoint process. These relations imply the modified FDRs (17) and (18) because $\langle h_1^{\rm R}\hat{h}_2^{\rm R}\tilde{\rangle} = 0$ for $t_1 > t_2$ (as this response function is causal). Note that these modified FDRs were derived above from $PS_{1,2}$, which are infinitesimal Grassmann SUSYs, in contrast to $\mathrm{TR}_{1,2}$, which are discrete symmetries. The mapping (25) also allows one to recover that $e^{-\mathcal{H}/T}$ is the steady state [75]. Comparing (26) and (27) to (15) and (16), we also identify the Grassmann correlator $\langle\overline{\Psi}_1\Psi_2\rangle^{\dagger}$ for $\mathbb{S}^{\dagger}_{\rm SUSY}$ as being equal to the time-reversed response function $\langle h_1^{\rm R}\hat{h}_2^{\rm R}\tilde{\rangle}$ in the adjoint dynamics. This allows one to relate such Grassmann correlators to physical correlation and response functions.

As we now show, this can be inferred from a BRST SUSY. One can check by direct computation that either of the time-reversal transformations $\mathrm{TR}_{1,2}$ yields

$$\mathbb{S}^{\dagger}[h,\hat{h},\Psi,\overline{\Psi}] = \tilde{S}_{\rm SUSY}[h^{\rm S},\hat{h}^{\rm S},-\overline{\Psi}^{\rm R},\Psi^{\rm R}] \tag{28}$$

(note the exchange of Ψ and $\overline{\Psi}$), where $\tilde{S}_{\rm SUSY}$ is the original SUSY action (7), but for the adjoint process. It possesses a BRST symmetry of the type (8), from which we infer that $\langle\overline{\Psi}_1\Psi_2\rangle^{\dagger} \overset{(28)}{=} -\langle\Psi_1^{\rm R}\overline{\Psi}_2^{\rm R}\tilde{\rangle} \overset{(14)}{=} \langle h_1^{\rm R}\hat{h}_2^{\rm R}\tilde{\rangle}$. Hence, $\langle\overline{\Psi}_1\Psi_2\rangle^{\dagger}$ is a (time-reversed) response function for the adjoint dynamics, as noted above. Equation (28) also implies identities for higher-order correlations of Ψ and $\overline{\Psi}$.

VII. CORRELATED NOISE

For noises correlated as $\langle\eta_i(t)\eta_j(t')\rangle = 2TD_{ij}\delta(t'-t)$ with a symmetric invertible matrix D, the previous results can be generalized as follows. Keeping the same definition for the quasipotential $\mathcal{H}$, the force is now decomposed as $f_i = D_{ij}(-\partial_j\mathcal{H} + g_j)$ instead of (3) and the stationary condition (4) becomes $\frac{1}{T}g_i D_{ij}\partial_j\mathcal{H} = D_{ij}\partial_i g_j$. The action $S_{\rm SUSY}$ is the same with $\hat{h}_i^2$ replaced by $\hat{h}_i D_{ij}\hat{h}_j$, and it verifies the BRST (8). Taking matrix notations and defining now $\tilde{\eta} = \partial_t h + D(\nabla\mathcal{H} + g)$ and $\bar{\eta} = -\partial_t h + D(\nabla\mathcal{H} + g)$, the actions

$$S^{\dagger}_{\rm SUSY} = \int_t \left\{ \hat{h}(\eta - TD\hat{h}) + \frac{1}{T}gD\nabla\mathcal{H} - \overline{\Psi}\,\tilde{\eta}'^{\dagger}\Psi \right\},$$

$$\mathbb{S}^{\dagger}_{\rm SUSY} = \int_t \left\{ \hat{h}(\eta - TD\hat{h}) - \frac{1}{T}(\partial_t h - Dg)\nabla\mathcal{H} + \Psi\bar{\eta}'\overline{\Psi} \right\}$$

generalize (9) and (19), and a factorized form similar to (21) can be identified [75]. SUSYs $PS_{1,2}$ become [87]

$$\mathrm{PS}_1:\ \begin{cases} \delta h = \varepsilon T\overline{\Psi}, & \delta\hat{h} = \varepsilon D^{-1}(\partial_t h - Dg[h])'\overline{\Psi}, \\ \delta\Psi = \varepsilon D^{-1}(\partial_t h - Dg[h] - TD\hat{h}), & \delta\overline{\Psi} = 0, \end{cases}$$

$$\mathrm{PS}_2:\ \begin{cases} \delta h = \varepsilon T\overline{\Psi}, & \delta\hat{h} = \varepsilon(\nabla\mathcal{H}[h])'\overline{\Psi}, \\ \delta\Psi = -\varepsilon D^{-1}(D\nabla\mathcal{H}[h] - TD\hat{h}), & \delta\overline{\Psi} = 0, \end{cases}$$

and they imply the following modified FDR:

$$\langle h_1(\partial_t h_2 - Dg_2[h_2])\rangle = T\langle h_1 D\hat{h}_2\rangle - T\langle\overline{\Psi}_1 D\Psi_2\rangle^{\dagger}, \tag{29}$$

$$\langle h_1\nabla\mathcal{H}[h_2]\rangle = T\langle h_1\hat{h}_2\rangle + T\langle\overline{\Psi}_1\Psi_2\rangle^{\dagger}. \tag{30}$$

One has $\langle\overline{\Psi}_1\Psi_2\rangle^{\dagger} = \langle\overline{\Psi}_1 D\Psi_2\rangle^{\dagger} = 0$ if $t_1 > t_2$.

VIII. KPZ EQUATION AND CONTINUOUS SPACE

Choosing $\mathcal{H}[h] = \frac{\nu}{2}\sum_i(\nabla_i h)^2$ (with $\nabla_i h = h_{i+1} - h_i$) and $g_i[h] = \frac{\lambda}{6}[(\nabla_i h)^2 + \nabla_i h\nabla_{i-1}h + (\nabla_{i-1}h)^2]$, the Langevin equation (1) is a discretized version of the continuum KPZ equation $\partial_t h = \nu\partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + \eta$. It possesses the SUSYs that we have derived together with the modified FDRs, since the chosen discretizations of $\mathcal{H}$ and of the nonlinear term $g_i[h]$ ensure that both sides of the stationary condition (4) are 0. Such a situation with an orthogonal decomposition of the force ($g_i\partial_i\mathcal{H} = 0$) and a zero divergence ($\partial_i g_i = 0$) could be generic [51].

If i is a lattice index, the continuous-space limit of our results is obtained directly. For KPZ, one has, for instance,

$\langle h_1(\partial_t h - \frac{\lambda}{2}(\partial_x h)^2)_2\rangle = T\langle h_1\hat{h}_2\rangle$ and $\langle h_1\partial_x^2 h_2\rangle = T\langle h_1\hat{h}_2\rangle$ if $t_1 > t_2$. The second relation was derived in [84]. Note that *not all spatial discretizations of the nonlinear term satisfy* (4): hence, in general, the discretization of gradients must be specified when it comes to SUSY, FDR, and time reversal because $\partial_i g_i$ is ambiguous in the continuum if $g[h]$ depends on gradients—as also seen in singularities of the functional Fokker-Planck equation [75].

IX. DISCUSSION AND OUTLOOK

We have identified SUSYs related to arbitrary Langevin equations with Gaussian additive white noise, generalizing long-known results that were restricted to reversible settings [11,12]. They can be expressed both in the MSRJD formalism and in the Onsager-Machlup one. The price to pay is an explicit dependency on the stationary state, and a more complex structure: two actions both representing the same physical process and each presenting different SUSYs. The important outcome is that they entail modified nonequilibrium FDRs [38] (that provide information on the steady state when it is not known).

As illustrated for the KPZ equation, the case of spatially continuous models is obtained directly from the results we presented, but the spatial discretization of gradients has to be specified (to make sense of $\partial_i g_i$ in the continuum).

The construction we presented is reminiscent of the derivation of the Jarzynski relation by Mallick *et al.* [88], and it would be interesting to find a unified framework. Our results apply to nonequilibrium models with a known steady state, such as the zero-range process [89–91] or mass transport models [92], and other cases [93–95]. In the small-noise limit, the adjoint dynamics is often known in macroscopic fluctuation theory [96], and thus the SUSYs $PS_{1,2}$ should be applicable. We note in general that in the small-noise asymptotics of the Langevin process [97], the quasipotential $\mathcal{H}[h]$ can become a singular (nondifferentiable) function of its argument [98–100], even though $\mathcal{H}[h]$ is regular as long as T is finite. This implies that the $T \to 0$ limit has to be taken in a careful way. The case of non-Gaussian noise could be investigated [28]. The extensions to inertial Langevin equations, singular (D not invertible) or colored noise, or multiplicative noise deserve further investigation.

The SUSYs that we have unveiled are defined for path integrals, but the reversible SUSY also has an operator version, with the Fokker-Planck operator completed by fermionic operators representing the Grassmann variables. It was used by Kurchan *et al.* to study metastability in overdamped [25] and inertial [26] Langevin dynamics; see, also, [101]. It would be interesting to translate our results in these settings. It is a nontrivial task already in the overdamped case since in the reversible case the equality of the actions (7) and (9) corresponds to the fact that the extended (fermionic) Fokker-Planck operator can be made Hermitian (which is an essential aspect of Kurchan *et al.*'s construction), while the same property does not hold in the generic irreversible case that we are considering. Last, it could be instructive to identify the relation between our results and the slave process of Refs. [101,102], and, more generally, with cohomology [103,104].

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$\varepsilon\Psi] = \varphi_i[h] + \varepsilon\varphi_i[h]'\Psi$. One has, for instance, for $\eta_i[h]$ defined in Eq. (5): $\eta_i'[h]\Psi = (\delta_{ij}\partial_t - \partial_j f_i[h])\Psi_j$.

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[73] Denoting, by $X_t^{\rm s} = \frac{X_{t+\Delta t}+X_t}{2}$, the Stratonovich discretization, $\int_t \overline{\Psi}_i\eta_i'[h]\Psi = \int_t \overline{\Psi}_i(\delta_{ij}\partial_t - \partial_j f_i[h])\Psi_j$ must be discretized as $\sum_t \Delta t\,\overline{\Psi}_{i,t+\Delta t}(\frac{\Psi_{i,t+\Delta t}-\Psi_{i,t}}{\Delta t} - \partial_j f_i[h_t^{\rm s}]\Psi_{j,t}^{\rm s}) = \sum_{tt'}\overline{\Psi}_{i,t'}M_{i,t';j,t}\Psi_{j,t}$ with the matrix elements given by $M_{i,t';j,t} = \delta_{ij}(\delta_{t',t} - \delta_{t',t+\Delta t}) - \Delta t\,\partial_j f_i[h_{t'}]\frac{\delta_{t',t}+\delta_{t',t+\Delta t}}{2}$. As the Grassmann integral yields the determinant of M, and as M is triangular in the time coordinate, only the diagonal $t' = t$ matters and $\det M = \sum_t \det(\delta_{ij} - \Delta t\,\partial_j f_i[h])$. One thus recovers the Jacobian [72].

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