

Semiclassical treatment of spin system by means of coherent states

Hernán G. Solari

Citation: *J. Math. Phys.* **28**, 1097 (1987); doi: 10.1063/1.527554

View online: <http://dx.doi.org/10.1063/1.527554>

View Table of Contents: <http://jmp.aip.org/resource/1/JMAPAQ/v28/i5>

Published by the [American Institute of Physics](#).

Related Articles

Phase induced transport of a Brownian particle in a periodic potential in the presence of an external noise: A semiclassical treatment

J. Math. Phys. **52**, 073302 (2011)

Semiclassical initial value representation study of internal conversion rates

J. Chem. Phys. **134**, 234305 (2011)

Semiclassical propagator for SU(n) coherent states

J. Math. Phys. **52**, 052104 (2011)

Asymptotic expansion for the wave function in a one-dimensional model of inelastic interaction

J. Math. Phys. **52**, 022103 (2011)

Investigation of the time evolutions of some log-periodic oscillators

J. Math. Phys. **51**, 122901 (2010)

Additional information on *J. Math. Phys.*

Journal Homepage: <http://jmp.aip.org/>

Journal Information: http://jmp.aip.org/about/about_the_journal

Top downloads: http://jmp.aip.org/features/most_downloaded

Information for Authors: <http://jmp.aip.org/authors>

ADVERTISEMENT

**AIP**Advances

Submit Now

**Explore AIP's new
open-access journal**

- **Article-level metrics
now available**
- **Join the conversation!
Rate & comment on articles**

Semiclassical treatment of spin system by means of coherent states

Hernán G. Solari^{a)}

Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

(Received 4 November 1985; accepted for publication 7 January 1987)

The semiclassical time-dependent propagator is studied in terms of the SU(2) coherent states for spin systems. The first- and second-order terms are obtained by means of a detailed calculation. While the first-order term was established in the earlier days of coherent states the second-order one is a subject of contradiction. The present approach is developed through a polygonal expansion of the discontinuous paths that enter the path integral. The results here presented are in agreement with only one of the previous approaches, i.e., the one developed on Glauber's coherent states by means of a direct WKB approximation. It is shown that the present approach gives the exact result in a simple case where it is also possible to observe differences with previous works.

I. INTRODUCTION

The formulation of semiclassical approaches to quantum problems has received a renewed interest following the popularity of the so-called coherent states (CS's),^{1,2} which are in some sense the most classical states. Since the pioneering work by Klauder,³ where the path integral formulation was established, it was apparent that a so-called complexification of the (real) classical variables (essentially position and impulse) was necessary.³ This complexification obscured the derivation of the semiclassical time propagator (SP) especially when the second-order term, i.e., the reduced propagator (RP), ought to be considered. Attempts to avoid this procedure^{4,5} have resulted in a heuristic formulation of the SP in a P -form⁶ (P - and Q -forms associated with CS's were introduced from the beginning by Glauber⁷). Later work on Glauber's coherent states has shown⁸ that the RP in the P -form has a more complicated structure than the one first assumed.

The reduced propagator has also been a subject of controversy in the path integral approach. The imposition of continuity conditions⁹⁻¹¹ to the paths considered has forced the application of the path expansion procedure¹² for the evaluation of the RP, and as a result of this procedure, the second-order term has been formally expressed in terms of the eigenvalues of a Sturm-Liouville problem.^{9,10} This procedure has received some criticism¹³ because it produces incorrect behavior of the SP at the starting time, among other problems.¹³ Other attempts to find the semiclassical propagator present problems in the identification of the correct classical Hamiltonian.^{14,15}

In the present work, we study the semiclassical propagator for spin and quasispin systems using the following steps.

(i) We decompose the propagator by means of the Trotter product formulas and slip-in identities between the product terms as done in Refs. 3-6, 9-11, and 16, but, taking advantage of the coherent states' overcompleteness,^{1,2,7} the identities are taken in a more general form than in the previous works. This generality is not really a necessary tool, but it makes the following steps clearer.

(ii) We evaluate all the integrals by the Laplace method. This method requires the complexification of the variables, but, by virtue of the generality introduced in (i), this reduces to fixing the free complex parameter (which labels the equivalent identities) to a different value for each time.

(iii) The second-order term is evaluated directly from the Laplace method working out a second-order differential equation for the reduced propagator.

(iv) Finally, the equation for the reduced propagator is integrated.

The steps (i) and (ii) are developed in Sec. II; Sec. III is reserved for a detailed calculation of the reduced propagator [steps (iii) and (iv)] while Sec. IV is devoted to an almost trivial example which already shows the differences between this work and the previous ones. The conclusions and perspectives are presented in Sec. V.

II. SU(2) PATH-INTEGRAL-LIKE FORMULATION

A. The formulation

The path integral formulation can be easily found using the slip-in identities decomposed as addition of coherent states^{3-5,9-11} between the terms in the Trotter product formulas

$$U = \exp(-iHt) = \lim_{N \rightarrow \infty} (1 - iHt/N)^N. \quad (2.1)$$

The standard identity in terms of CS's is written^{1,2} as

$$I = \frac{2J+1}{i2\pi} \int_C |z\rangle \langle z| \frac{dz \wedge dz}{(1+zz^*)^2}, \quad (2.2)$$

where

$$|z\rangle = \exp(z'J_+ - z'^*J_-)|J, -J\rangle, \quad (2.3a)$$

$$z' = e^{-i\theta/2}, \quad z = e^{-i\theta} \tan(\theta/2), \quad (2.3b)$$

(J_+, J_-, J_z) are the three generators of the SU(2) group, and $|J, -J\rangle$ is the extremal state ($J_z|J, -J\rangle = -J|J, -J\rangle$) of the J -irreducible representation of SU(2), while the domain of integration C is the complex plane.

The coherent state (2.3) may also be written taking advantage of the BCH theorems¹ in the form

^{a)} Fellow of the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

$$|z\rangle = \exp(zJ_+) \exp(\ln(1 + zz^*)J_z) \exp(-z^*J_-) |J, -J\rangle, \quad (2.4a)$$

$$|z\rangle = \exp(zJ_+) |J, -J\rangle (1 + zz^*)^{-J}, \quad (2.4b)$$

$$|z\rangle = |z\rangle (1 + zz^*)^{-J}. \quad (2.4c)$$

This last formula [(2.4c)] defines the unnormalized coherent state (curved brackets) which allows us to rewrite the identity (2.2) in the form

$$I = \frac{2J+1}{2\pi i} \int_C |z\rangle \langle z| [(1 + zz^*)^2 (z|z)]^{-1} dz \wedge dz^*. \quad (2.5)$$

As the CS's form an overcomplete set of states there are many different ways of writing the identity; for instance, multiplying (2.5) by

$$I = \exp(\delta J_+) \exp(-\delta J_+),$$

we obtain

$$I = II = \exp(\delta J_+) \exp(-\delta J_+) I \\ = \exp(\delta J_+) I \exp(-\delta J_+),$$

$$I = \frac{2J+1}{2\pi i} \int_C \exp(\delta J_+) |z\rangle \langle z| \exp(-\delta J_+) \\ \times [(z|z)(1 + zz^*)^2]^{-1} dz \wedge dz^*,$$

which may be put in the form (applying again the disentangling theorems¹)

$$I = \frac{2J+1}{2\pi i} \int_D |y\rangle \langle x| [(x|y)(1 + yx^*)^2]^{-1} dy \wedge dx^*, \quad (2.6a)$$

where x^* and y depend on z and z^* in the following specific form:

$$y = z + \delta, \quad (2.6b)$$

$$x^* = z^*/(1 - z^*\delta). \quad (2.6c)$$

The domain of integration C in (2.5) transforms into

$$D = \{(y, x^*) \text{ such that } \\ (y - \delta)^* = x^*/(1 + x^*y)\} \text{ in (2.6).}$$

The identity (2.6) is valid for any arbitrary complex number δ , just because it does not depend on it.

Following the standard procedure^{4,5,9,10,16} we obtain the following expression for the matrix elements of the propagator between CS's:

$$\langle \phi | U | \psi \rangle = \lim_{N \rightarrow \infty} \int_{D_0} \cdots \int_{D_N} \prod_{n=0}^N dy_n \wedge dx_n^* \\ \times \{(2J+1) [2\pi i (1 + y_n x_n^*)^2]^{-1}\} \exp(F), \quad (2.7)$$

where F has the form

$$F = \sum_{n=1}^N \ln \left[\frac{\langle x_n | y_{n-1} \rangle}{\langle x_n | y_n \rangle} \right] - \frac{it}{N} \mathcal{H}(y_{n-1}, x_n^*)$$

$$+ \ln [\langle \phi | y_N \rangle \langle x_0 | \psi \rangle / \langle x_0 | y_0 \rangle] \quad (2.8)$$

and where the classical Hamiltonian $\mathcal{H}(y_{n-1}, x_n^*)$ reads

$$\mathcal{H}(y_{n-1}, x_n^*) = \langle x_n | H | y_{n-1} \rangle / \langle x_n | y_{n-1} \rangle. \quad (2.9)$$

At this point we note that there is no reason for requiring continuity of the paths just because we are dealing with nonorthogonal states. In this respect we recall that in earlier works on the subject^{3,9,11} only almost-everywhere continuous paths were considered. The contribution of the discontinuous paths can be determined by the following argument: considering the evaluation of the matrix elements of the identity (2.6),

$$\langle \phi | \psi \rangle = \frac{2J+1}{2\pi i} \int_D \frac{\langle \phi | y \rangle \langle x | \psi \rangle}{\langle x | y \rangle (1 + x^*y)^2} dy \wedge dx^*, \quad (2.10)$$

we observe that as long as the integrand is a nonsingular c -number all the allowed values of (y, x^*) contribute to the integral and not only just $y = \psi, x^* = \phi^*$ (it is even not necessarily in the domain of integration for an arbitrary δ !). Further, the integrand can never become singular, as is easily seen by inspection of (2.2).

The evolution operator has been decomposed, in our case, in an infinite product of infinitesimal steps (21). Each of the terms in the product is very like the identity but because of the argument concerning the matrix elements of the identity [cf. (2.10)] no notion of continuity of the paths follows from this observation. In fact the opposite is true. On the other hand, if the identities inserted between the terms of (2.1) were expressed in terms of δ -orthogonal states, an intuitive notion of continuity of the paths would follow.

In the following, we shall include discontinuous paths (as suggested in Ref. 13), with the understanding that the state $|y_n\rangle$ in (2.7) is not supposed to be $|y_n\rangle = |y_{n-1}\rangle + O(1/N)$. At this time we will not formulate a formal path integral which would call for the time derivatives of discontinuous paths. A discussion of the subject may be found in Ref. 16. In the semiclassical evaluation of (2.7) we follow a method which closely resembles the polygonal formulation of the path integral.¹³ We left the large N limit as the last step to be taken.

B. Classical evaluation of the integrals

The evaluation of integrals, which depends on complex arguments (but real variables) by the Laplace or saddle point methods, requires that the integration path be extended to the complex plane¹⁷; this procedure was called complexification by Klauder.³ In the present situation such a deformation of the integration path has already been done in (2.6) and the extremal points are identified by maximizing F [(2.8)] in all the variables, leading to the following set of equations:

$$\frac{\partial F}{\partial x_n^*} = 0 = \frac{\partial \{ \ln [\langle x_n | y_{n-1} \rangle / \langle x_n | y_n \rangle] - it/N \mathcal{H}(y_{n-1}, x_n^*) \}}{\partial x_n^*}, \quad n = 1, \dots, N, \quad (2.11a)$$

$$\frac{\partial F}{\partial x_0^*} = 0 = \frac{\partial \ln \{ \langle x_0 | \psi \rangle / \langle x_0 | y_0 \rangle \}}{\partial x_0^*}, \quad \text{i.e., } \psi = y_0, \quad (2.11b)$$

the “extra-phase” term because it provides an extra phase term in the simplest examples, although it is not necessarily of modulo one in general. We do not have at present a physical interpretation for this term, but we observe that it is like a signature of the coherent states, in the sense that the expression (3.15) looks like the expression of the semiclassical propagator in term of space coordinates¹³ except for the presence of this term, (3.18), and the ratio of the metric at the initial and final points in (3.17). An equivalent term to (3.18) is also present in the P -propagator of Ref. 8.

IV. EXAMPLE

The simplest example we may look for is the J_z Hamiltonian:

$$H = J_z. \quad (4.1)$$

The classical Hamiltonian (2.9) is now

$$\mathcal{H}(y, x^*) = (x|J_z|y)/(x|y) = -J(1 - yx^*)/(1 + yx^*) \quad (4.2)$$

and the classical motion (3.16) is

$$i\dot{y} = y, \quad y(0) = \psi, \quad (4.3a)$$

$$i\dot{x}^* = -x^*, \quad x^*(t) = \phi^*. \quad (4.3b)$$

These equations have straightforward integrals that allow the evaluation of the action in the form given by (3.2):

$$S(\psi, \phi^*, t) = Jt - i2J \ln(1 + \psi\phi^*e^{-it}). \quad (4.4)$$

Taking the second derivative of S (4.4) we obtain

$$\frac{\partial^2 S}{\partial \psi \partial \phi^*} = - \frac{i2Je^{-it}}{(1 + \psi\phi^*e^{-it})^2}. \quad (4.5)$$

The evaluation of the SP (3.15) from (4.3)–(4.5) gives

$$(\phi|U|\psi) = e^{iJt}(1 + \psi\phi^*e^{-it})^{2J}. \quad (4.6)$$

This last formula [(4.6)] is in fact exact for the matrix elements of the SP. The result (4.6) in this example is more accurate than the previous results where the same matrix elements were calculated in the form [Eqs. (3.7) and (3.49), Ref. 9]

$$(\phi|U|\psi) = \exp(iS(\psi, \phi^*, t))(2i\pi \sin(t))^{-1/2}. \quad (4.7)$$

Before attempting a comparison between the present results and the corresponding ones of Refs. 8, 14, and 15, we have to go from SU(2) coherent states to $Np4$ coherent states (Gaussian wave packets in the space or momentum basis). The procedure consists of contracting the SU(2) algebra into the $Np4$ algebra and simultaneously mapping the coherent states. The details of the method are explained in detail in Refs. 1 and 18. We recall here the main results.

Under the limit $J \rightarrow \infty$ the operators and coherent states associated with SU(2) go into operators and coherent states of $Np4$ in the form

$$\begin{aligned} \text{SU}(2) &\rightarrow Np4, \\ J_z + J &\rightarrow \hat{N} = a^+ a \quad (\text{number operator}), \\ J_+ / (2J)^{1/2} &\rightarrow a^+ \quad (\text{creation operator}), \\ J_- / (2J)^{1/2} &\rightarrow a \quad (\text{destruction operator}), \end{aligned}$$

$$\begin{aligned} |J, -J\rangle &\rightarrow |0\rangle \quad (\hat{N}|0\rangle = 0|0\rangle), \\ (2J)^{1/2} y &\rightarrow y \quad (\text{coherent states map}). \end{aligned} \quad (4.8)$$

The contraction can be seen in essence as the linear expansion of the phase space $\{y, x^*\}$ around the point $\{0, 0\}$.

With these identifications, we obtain, from (4.6), the matrix elements of $F = \exp(-it\hat{N})$ expressed as

$$(\phi|F|\psi) = \lim_{J \rightarrow \infty} (\phi|U|\psi)e^{-iJt} = \exp(\psi\phi^*e^{-it}), \quad (4.9)$$

where the limit will be understood as the contraction procedure previously outlined.

This latest expression (4.9) is exact and is the one obtained in Ref. 8, while it appears in Refs. 14 and 15 multiplied by $e^{it/2}$. This factor is irrelevant in the present trivial example but it accounts for a missing term in the general semiclassical expression of Ref. 14. (In Ref. 15 the factor was compensated by an *ad hoc* identification of the classical Hamiltonian.)

The contraction procedure applied here is not limited to the Hamiltonian of the example and is valid in general.

V. CONCLUSIONS

We have developed the semiclassical propagator in terms of SU(2) coherent states in an almost closed form. The resulting formula is well behaved for short times and in addition it matches the exact result for Hamiltonians which are linear combinations of the SU(2) generators. It also agrees with the results obtained in Ref. 8 using Glauber's coherent states in a direct WKB approximation to the SP in the P -form.

Looking for possible generalizations, we recall here that the present approach is fully based on the existence of an algebraic classical limit,¹⁹ expressed by the large $(2J)$ approximation. *While it does not appear that it could be difficult to generalize these results to other systems from a technical point of view, it is worth keeping in mind that the existence of an algebraic limit is a requisite from both physical and mathematical points of view.* (It may express the existence of a large number of particles or quasiparticles or to have other meaning depending upon the problem.)

In order to make sense the evaluation of the integrals by the Laplace or saddle point methods it is required that the overlap between two unnormalized coherent states behaves as C^λ , where C is a complex number and λ is the order parameter that is expected to be linked with physical situations. The nonexistence of a parameter in which the asymptotic expansion is carried out makes the application of the Laplace method uncertain and does not make room for necessary operations like the one performed while going from (3.6) to (3.7). As a major mention of the importance of this fact we recall that the standard time-dependent Hartree–Fock equations, which have been formally derived in the classical limit,^{4,11} do not have an identified large parameter associated.²⁰ This fact raises important questions about the justification of these derivations.

As a physical situation that may be treated by the present approach we may mention the Coulomb excitation of a

nucleus as the result of scattering if the nucleus is described by an IBM model.²⁰ Another point of possible physical interest is the requantification of the solutions applying Gutzwiller's method²² adapted to CS's. In this context, the lowest lying state in this approximation is the one predicted by the random phase approximation as it may be easily realized shifting the real time to an imaginary one ($i\beta$) and looking for the $\beta \rightarrow \infty$ limit (i.e., the zero temperature limit).

Our last point about the present approach is that it does respect dynamical symmetries if they can be expressed by the exponential of a linear combination of the SU(2) generators.⁸ This point brings up several questions as to the correct way of taking mean values of operators in the semiclassical approximation because SU(2)-TDHF expressions are symmetry breaking (see, for example, Refs. 23 and 24). Further work on this subject is in progress.

ACKNOWLEDGMENTS

I fully acknowledge support of the Consejo Nacional de Investigaciones Cientificas y Tecnicas of Argentina and hospitality of Drexel University. I am kindly indebted to Dr. Michel Vallieres for a careful reading of previous versions of this work.

This work has been partially supported by the National

Science Foundation (NSF) under Grant No. PHY-84-41891.

- ¹R. Gilmore, *Lie Groups and Lie Algebras and Some of Their Applications* (Wiley, New York, 1974).
- ²A. Perelomov, *Commun. Math. Phys.* **26**, 222 (1972).
- ³J. Klauder, *Phys. Rev. D* **19**, 2349 (1979).
- ⁴H. Kuratsuji and T. Suzuki, *Phys. Lett. B* **92**, 19 (1980).
- ⁵H. Kuratsuji, *Phys. Lett. B* **103**, 79 (1981).
- ⁶T. Suzuki, *Nucl. Phys. A* **398**, 557 (1983).
- ⁷R. J. Glauber, *Phys. Rev.* **131**, 2766 (1963).
- ⁸H. G. Solari, *J. Math. Phys.* **27**, 1351 (1986).
- ⁹H. Kuratsuji and Y. Mizobuchi, *J. Math. Phys.* **22**, 757 (1981).
- ¹⁰H. Kuratsuji and Y. Mizobuchi, *Phys. Lett. A* **82**, 2798 (1981).
- ¹¹J. P. Blaizot and H. Orland, *Phys. Rev. C* **24**, 1740 (1981).
- ¹²S. Levit, *Ann. Phys. (NY)* **103**, 198 (1977).
- ¹³L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).
- ¹⁴Y. Weissman, *J. Chem. Phys.* **76**, 4067 (1982).
- ¹⁵Y. Weissman, *J. Phys. A: Math. Gen.* **16**, 2693 (1983).
- ¹⁶J. R. Klauder and B.-S. Skagerstan, *Coherent States* (World Scientific, Singapore, 1985).
- ¹⁷E. T. Copson, *Asymptotic Expansions* (Cambridge U.P., Cambridge, 1965).
- ¹⁸F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, *Phys. Rev. A* **6**, 2211 (1972).
- ¹⁹R. Gilmore, *J. Math. Phys.* **20**, 891 (1979).
- ²⁰S. Levit, *Phys. Rev. C* **21**, 1594 (1980).
- ²¹R. Gilmore and D. H. Feng, *Interacting Bose-Fermi Systems in Nuclei*, edited by F. Iachello (Plenum, New York, 1981).
- ²²M. C. Gutzwiller, *J. Math. Phys.* **12**, 343 (1971).
- ²³H. G. Solari and E. S. Hernandez, *Phys. Rev. C* **26**, 2310 (1982).
- ²⁴H. G. Solari and E. S. Hernandez, *Phys. Rev. C* **28**, 2472 (1983).