Population dynamics and rare events

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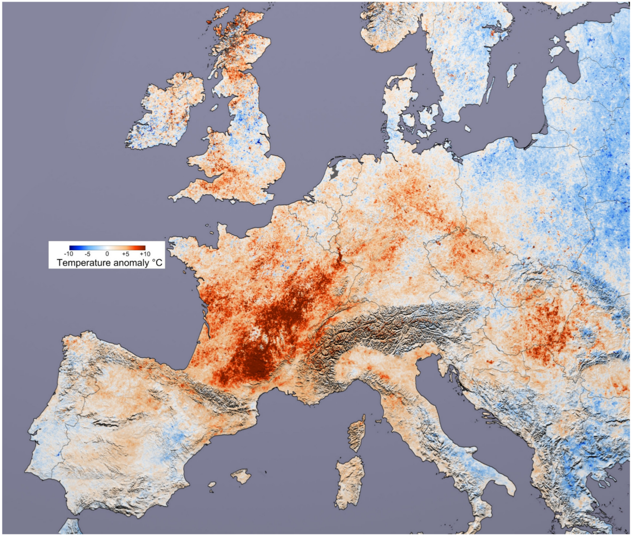
Why studying rare events?



2003 heat wave, Europe [Terra MODIS]

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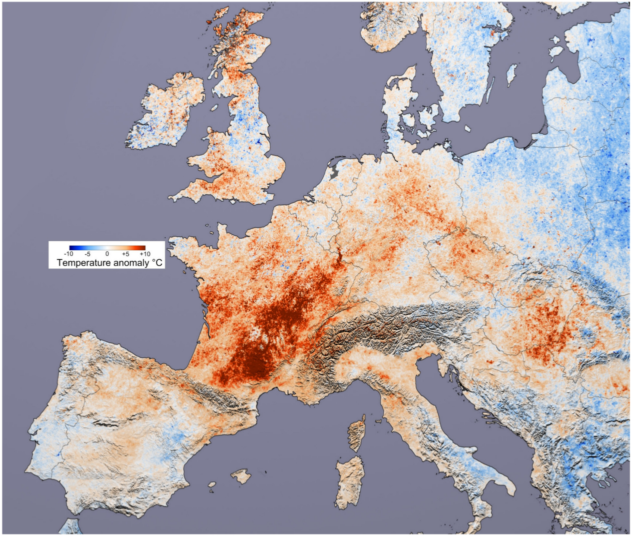

[Anomaly for **1-month** average] 2003 heat wave, Europe [Terra MODIS]

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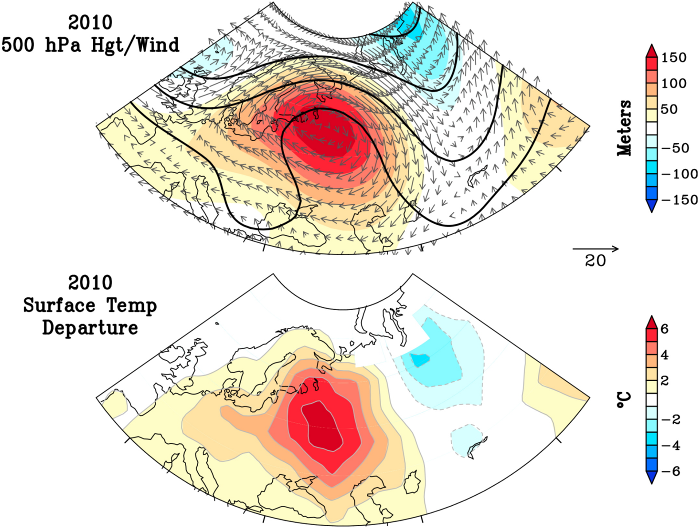

2010 heat wave in Western Russia [Dole *et al.*, 2011]

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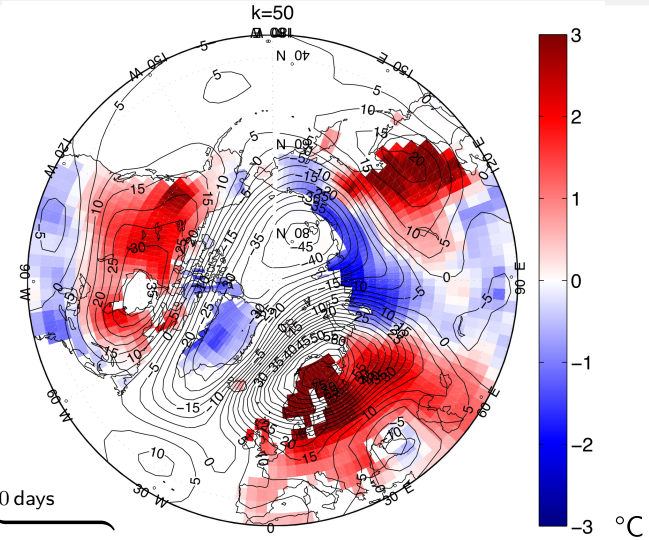

$$\overbrace{\frac{1}{t_{\max}}\int_0^{t_{\max}} dt\,\Delta T(t)}^{t_{\max}=40\text{ days}} > 2^\circ C \Rightarrow$$ « Teleconnection patterns » [Bouchet *et al.*]

Why studying rare events?

Questions for physicists and mathematicians:

- Probability and **dynamics** of rare events?
- How to **sample** these in numerical modelisation?
- Numerical **tools and methods** to understand their formation?

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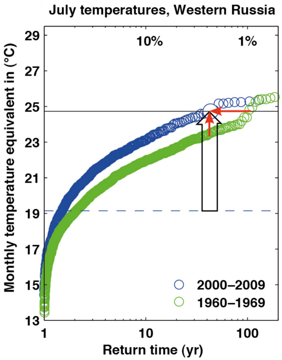


Evolution of the return time
of the monthly averaged temperature

$$\frac{1}{t_{\max}} \int_0^{t_{\max}} dt\ T(t)$$

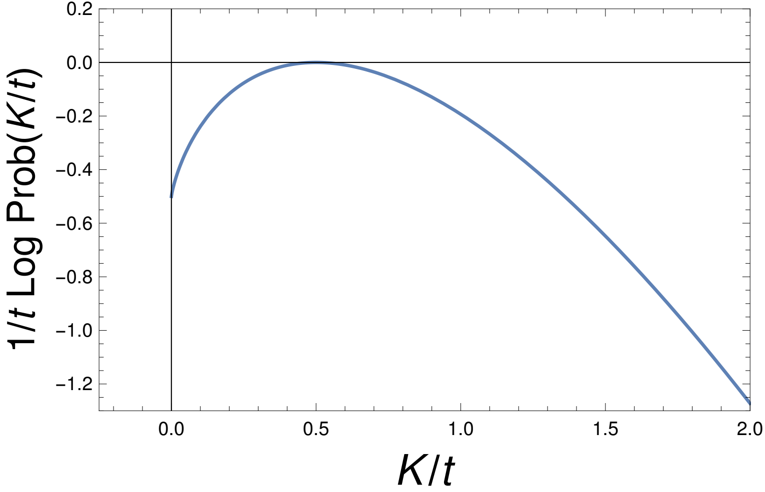
$\longleftrightarrow$ anthropogenic impact on climate?

[Otto *et al.*, 2012]

Outline

- Introduction
- Settings
- Different averages
- Feedback method
- Finite-time and finite-population scalings
- Open questions

Distribution of a time-extensive observable K on $[0, t]$



$$\mathrm{Prob}[K, t] \sim e^{t\,\varphi(K/t)}$$

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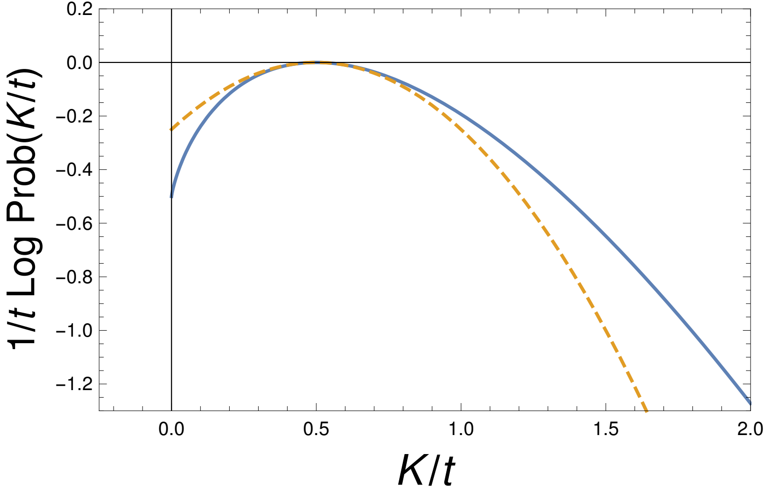

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s-modified dynamics

- Markov processes: Configs. $\mathcal{C}$, jump rates $W(\mathcal{C} \to \mathcal{C}')$

$$\partial_t P(\mathcal{C}, t) = \sum_{\mathcal{C}'} \Big\{ \underbrace{W(\mathcal{C}' \to \mathcal{C}) P(\mathcal{C}', t)}_{\text{gain term}} - \underbrace{W(\mathcal{C} \to \mathcal{C}') P(\mathcal{C}, t)}_{\text{loss term}} \Big\}$$

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- More detailed dynamics for $P(\mathcal{C}, K, t)$:

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s-modified dynamics $\quad K = k_{\mathcal{C}_0\mathcal{C}_1} + k_{\mathcal{C}_1\mathcal{C}_2} + \ldots$

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Numerical method [JB Anderson; D Aldous; P Grassberger; P Del Moral; ...]

Evaluation of large deviation functions [*à la* "Diffusion Monte-Carlo"]

$$\sum_{\mathcal{C}} \hat{P}(\mathcal{C}, s, t) = \left\langle e^{-s\,K} \right\rangle \sim e^{t\,\psi(s)} \qquad (\psi(s) = \text{CGF})$$

- discrete time: Giardinà, Kurchan, Peliti [PRL **96**, 120603 (2006)]
- continuous time: VL, Tailleur [JSTAT P03004 (2007)]

Cloning dynamics

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How to take into account loss/gain of probability?

- handle a large number N_c of copies of the system
- implement a selection rule: on a time interval Δt
  a copy in config $\mathcal{C}$ is replaced by $Y = \mathrm{e}^{\Delta t\, \delta r_s(\mathcal{C})}$ copies
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- optionally: keep population constant by non-biased pruning/cloning

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CGF estimator: $\psi(s) = \mathbb{E}\Psi(s)$ with $\Psi(s) = \log \underbrace{\prod_t \frac{N_c + Y_t - 1}{N_c}}_{\text{reconstituted population}}$

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Biological interpretation

- copy in configuration $\mathcal{C} \equiv$ organism of **genome** $\mathcal{C}$
- dynamics of rates $W_s \equiv$ **mutations**
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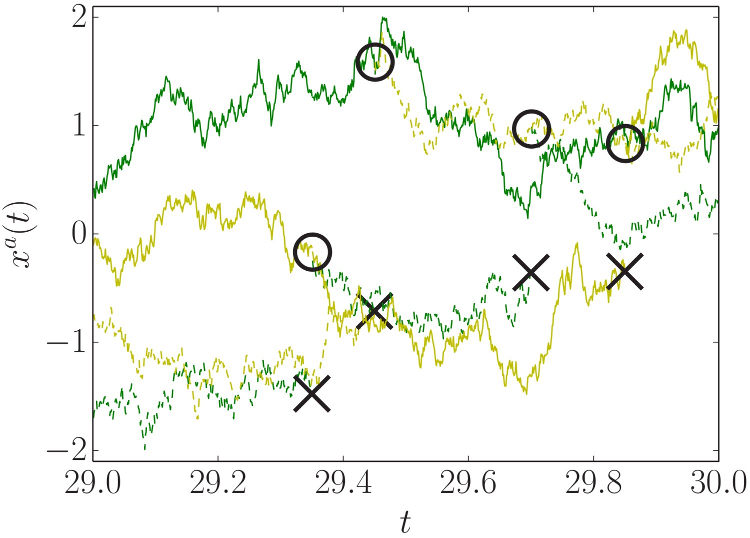
- the CGF $\psi(s)$ is a measure of the **fitness** of the population

An example: 4 copies, 1 degree of freedom $\mathcal{C} = x \in \mathbb{R}$



How to perform averages? (i) [spectral analysis]

⋆ Final-time distribution: *proportion* of copies in $\mathcal{C}$ at t

$$\langle N_{\rm nc}(t)\rangle_s$$

$$\langle N_{\rm nc}(\mathcal{C},t)\rangle_s$$

$$p_{\rm end}(\mathcal{C},t) = \frac{\langle N_{\rm nc}(\mathcal{C},t)\rangle_s}{\langle N_{\rm nc}(t)\rangle_s}$$

$[N_{\rm nc} =$ number in non-constant population dynamics$]$

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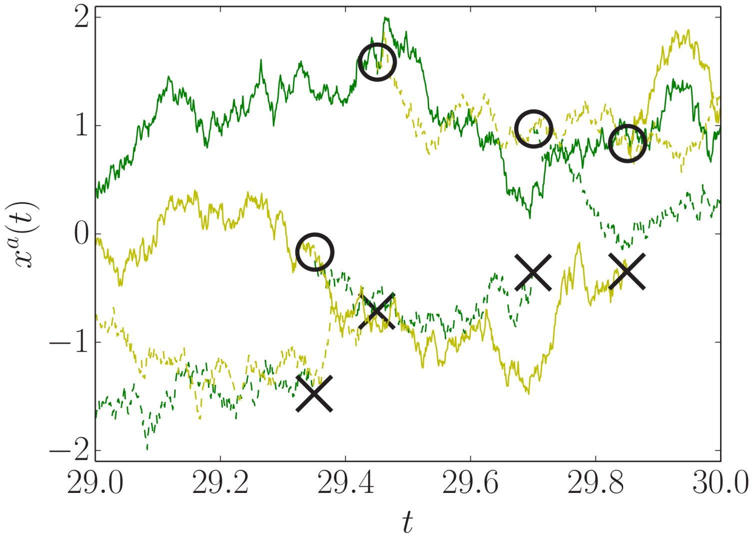
[N_{nc} = number in non-constant population dynamics]

Final-time distribution governed by **right** eigenvector.

An example: 4 copies, 1 degree of freedom $\mathcal{C} = x \in \mathbb{R}$



How to perform averages? (ii) Intermediate times

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Mid-time distribution governed by **left** and **right** eigenvectors.

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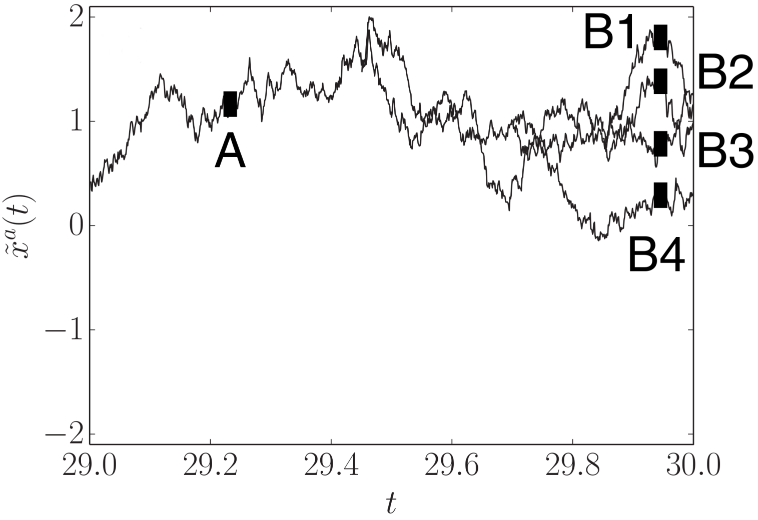


Huge sampling issue

How to perform averages?

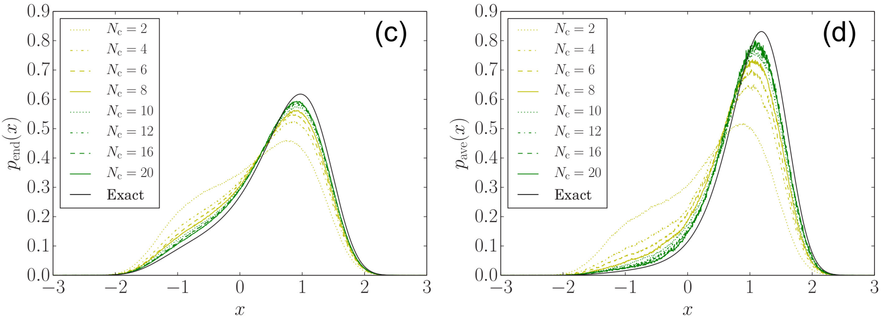
- ⋆ Mid-time ancestor distribution:

fraction of copies (at time t_1) which were in configuration $\mathcal{C}$, knowing that there are in configuration $\mathcal{C}_\mathsf{f}$ at final time t_f:

$$p_{\mathsf{anc}}(\mathcal{C}, t_1; \mathcal{C}_\mathsf{f}, t_\mathsf{f}) = \frac{\langle N_{\mathsf{nc}}(\mathcal{C}_\mathsf{f}, t_\mathsf{f} | \mathcal{C}, t_1)\rangle_s}{\sum_{\mathcal{C}'}\langle N_{\mathsf{nc}}(\mathcal{C}_\mathsf{f}, t_\mathsf{f} | \mathcal{C}', t_1)\rangle_s} \underset{t_{\mathsf{f},1}\to\infty}{\sim} \langle L|\mathcal{C}\rangle\langle\mathcal{C}|R\rangle = p_{\mathsf{ave}}(\mathcal{C})$$

The “ancestor statistics” of a configuration $\mathcal{C}_\mathsf{f}$ is thus independent (far enough in the past) of the configuration $\mathcal{C}_\mathsf{f}$.

Example distributions for a simple Langevin dynamics

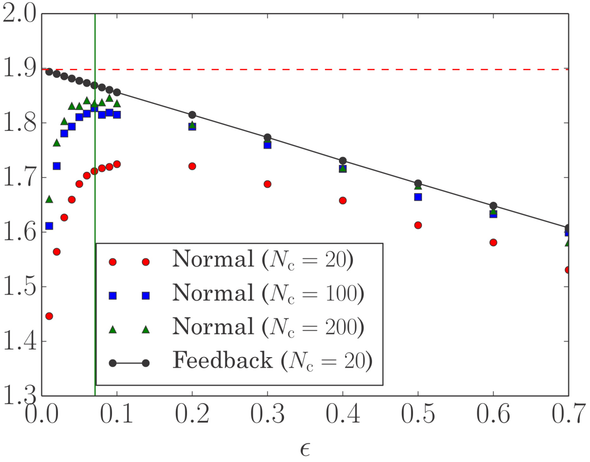


final-time: $p_{\mathsf{end}}(x)$
$(= R(x))$

intermediate-time: $p_{\mathsf{ave}}(x)$
$(= R(x)L(x))$

The small-noise crisis: systematic errors grow as $\epsilon \to 0$

CGF as a function of the noise amplitude ϵ:



Cause: as $\epsilon \to 0$, $p_{\text{ave}}(x)$ & $p_{\text{end}}(x) \to$ sharply peaked at *different points*

i.e. the clones do not sample correctly the phase space

The feedback method

How to make mid- and final-time distribution closer?

Driven/auxiliary dynamics: [Maes, Jack&Sollich, Touchette&Chetrite]

- Probability preserving
- No mismatch between p_{ave} and p_{end}
- Constructed as

$$\mathbb{W}_s^{\text{aux}} = L\mathbb{W}_s L^{-1} - \psi(s)\mathbf{1}$$

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$$\mathbb{W}_s^{\text{aux}} = L\mathbb{W}_s L^{-1} - \psi(s)\mathbf{1}$$

- Issue: determining L is difficult
- Solution: evaluate L as L_{test} on the fly **[feedback]** and simulate

$$\mathbb{W}_s^{\text{test}} = L_{\text{test}}\mathbb{W}_s L_{\text{test}}^{-1} \qquad \text{(induces \textbf{\textit{effective forces}})}$$

- **Iterate.** [For any L_{test}, the simulation is in principle correct.]

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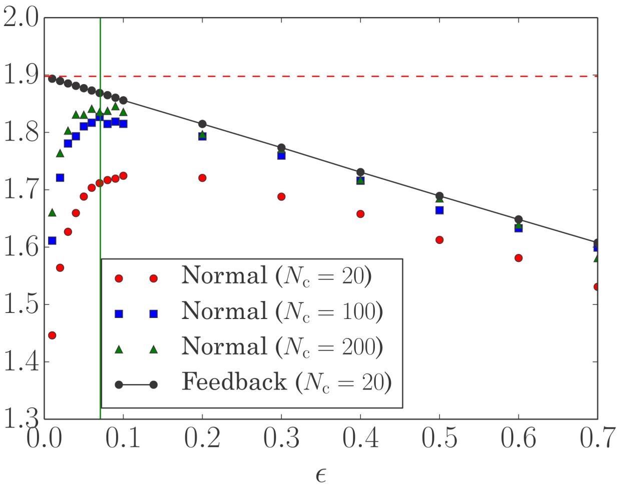
- **Iterate.** [For any L_{test}, the simulation is in principle correct.]

Similar in spirit to **multi-canonical** (*e.g.* Wang–Landau) approaches in static thermodynamics. [Here, one flattens the left-eigenvector of $\mathbb{W}_s^{\text{test}}$.]

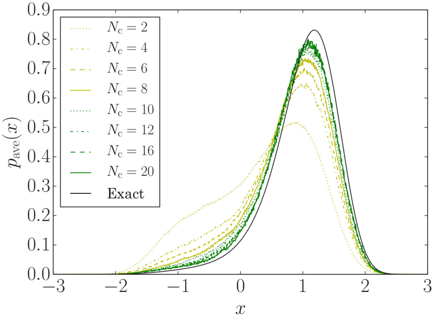
Improvement of the small-noise crisis (i.i)

CGF as a function of the noise amplitude ϵ:

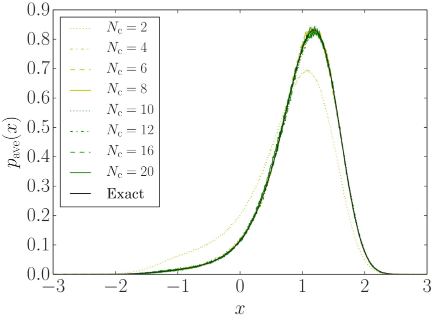


Physical insight: probability loss transformed into *effective forces*.

Improvement of the small-noise crisis (i.ii)



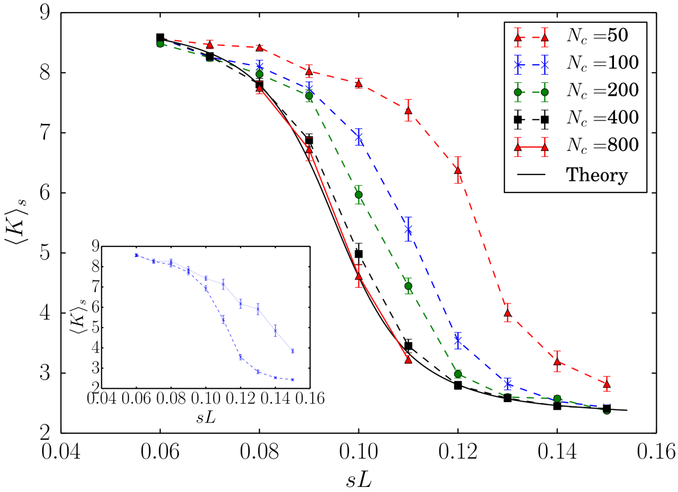
Without feedback



With feedback

Much more efficient evaluation of the biased distribution.
Even for a very crude (polynomial) approximation of the effective force.

Improvement of the small-noise crisis (ii)

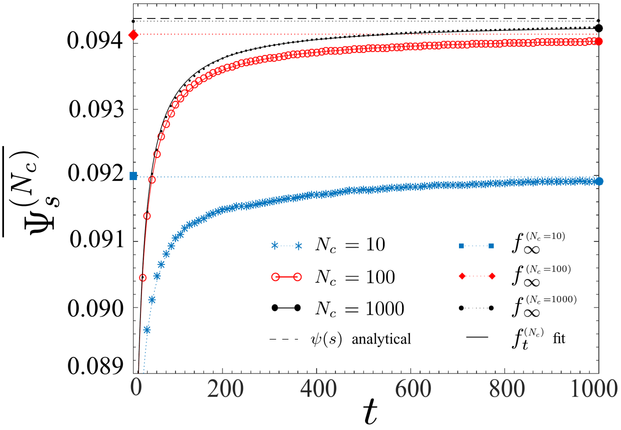


Interacting system in 1D.

Effective force: 1-, 2-, 3- body interactions only [also crude approx.].

Finite-time and -population effects

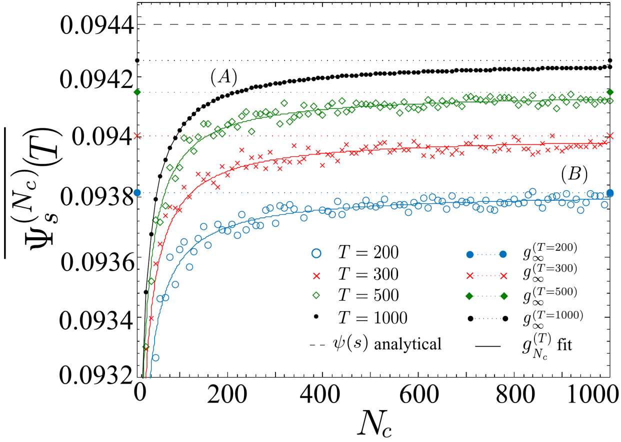
Finite-time scaling [fixed population N_c]



Estimator converges in $1/t$ to its infinite-time limit

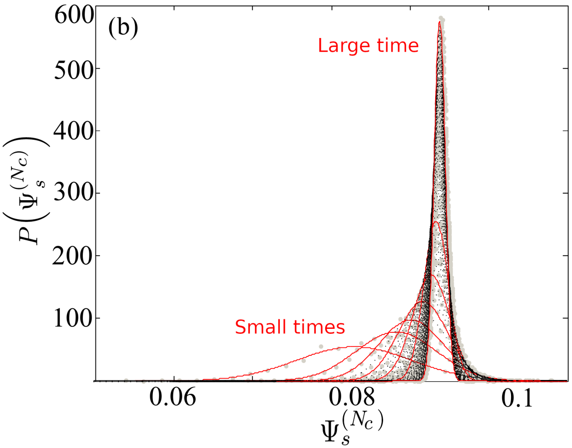
Understanding: the estimator is an additive observable of the pop. dyn.

Finite-N_c scaling **[fixed time]**



Estimator converges in $1/N_c$ to its infinite-population limit
Understanding: large N_c expansion, small-noise description

Distribution of the CGF estimator **[fixed population N_c]**



In the numerics: $\approx$ Gaussian when finite-N_c scaling is $O(1/N_c)$
A way to check why one is / is not in that regime

Summary and open questions (1)

Feedback method [with F Bouchet, R Jack, T Nemoto]

- Sampling problem (depletion of ancestors)
- On-the-fly evaluated auxiliary dynamics
- Solution to the small-noise crisis
- Systems with large number of degrees of freedom

Summary and open questions (1)

Feedback method [with F Bouchet, R Jack, T Nemoto]

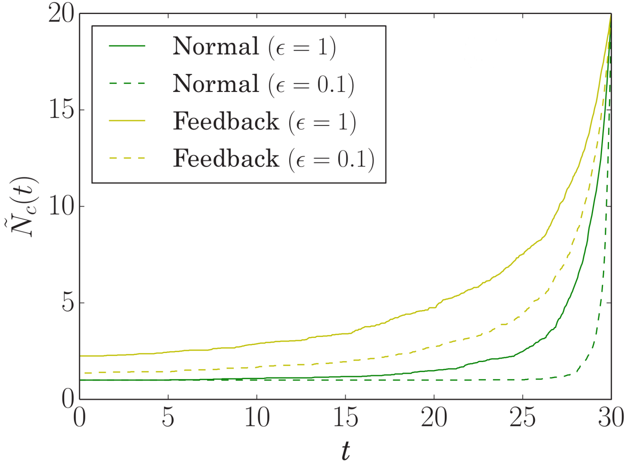
- Sampling problem (depletion of ancestors)
- On-the-fly evaluated auxiliary dynamics
- Solution to the small-noise crisis
- Systems with large number of degrees of freedom

Finite-population effects [with E Guevara, T Nemoto]

- Quantitative finite-N_{clones} scaling $\rightarrow$ interpolation method
- Initial transient regime due to small population
- Analogy with biology: many small islands *vs.* few large islands?
- Question: effective forces $\leftarrow$ selection?
- Question: relation to recent work of Ferré and Touchette?

Open questions (2): why is the feedback working?

Improvement of the depletion-of-ancestors problem:



Dashed line: lower noise Continuous line: higher noise

Open questions (3)

Finite-population and -time scalings

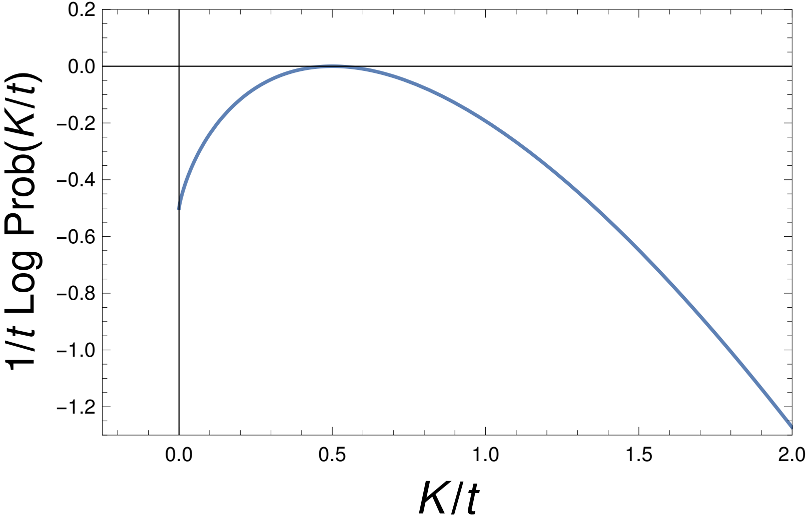
- Anomalous fluctuations (invalid $1/N_c$ asymptotics)
- Correct description of the meta-dynamics?
- Finite-N_c and -t scaling **with feedback**
- Phase transition in the distribution of the CGF estimator?

Thanks for your attention!

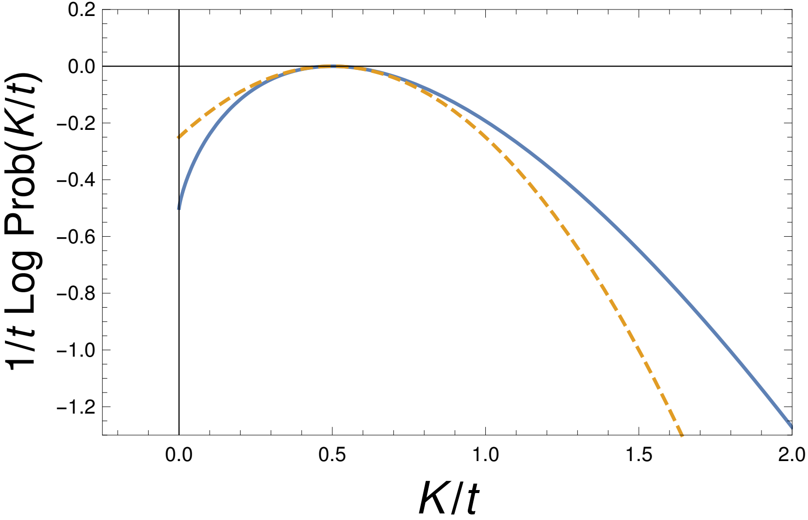
References:

- *Population dynamics method with a multi-canonical feedback control*
  Takahiro Nemoto, Freddy Bouchet, Robert L. Jack and Vivien Lecomte
  PRE **93** 062123 (2016)
- *Finite-time and finite-size scalings in the evaluation of large deviation functions: Analytical study using a birth-death process*
  Takahiro Nemoto, Esteban Guevara Hidalgo and Vivien Lecomte
  PRE **95** 012102 (2017)
- *Finite-size scaling of a first-order dynamical phase transition: adaptive population dynamics and effective model*
  Takahiro Nemoto, Robert L. Jack and Vivien Lecomte
  PRL **118** 115702 (2017)
- *Finite-time and finite-size scalings in the evaluation of large deviation functions: Numerical approach in continuous time*
  Esteban Guevara Hidalgo, Takahiro Nemoto and Vivien Lecomte
  PRE **95** 062134 (2017)

Supplementary material



$\mathrm{Prob}[K] \sim e^{t\,\varphi(K/t)}$



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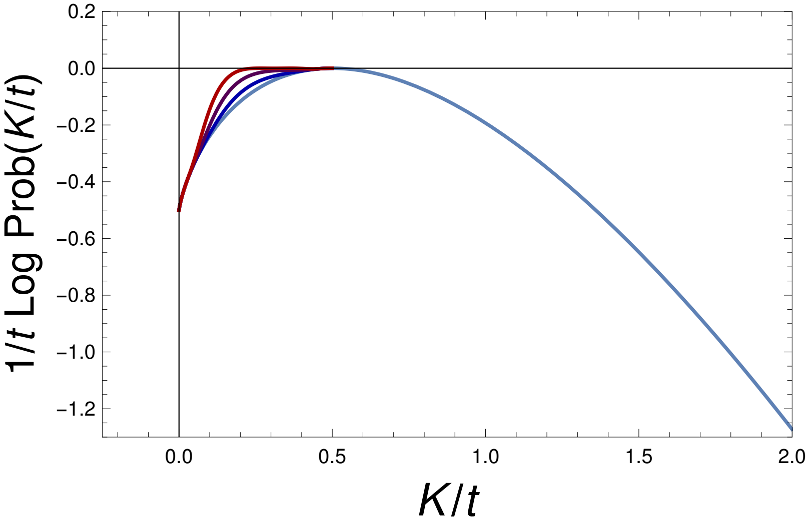

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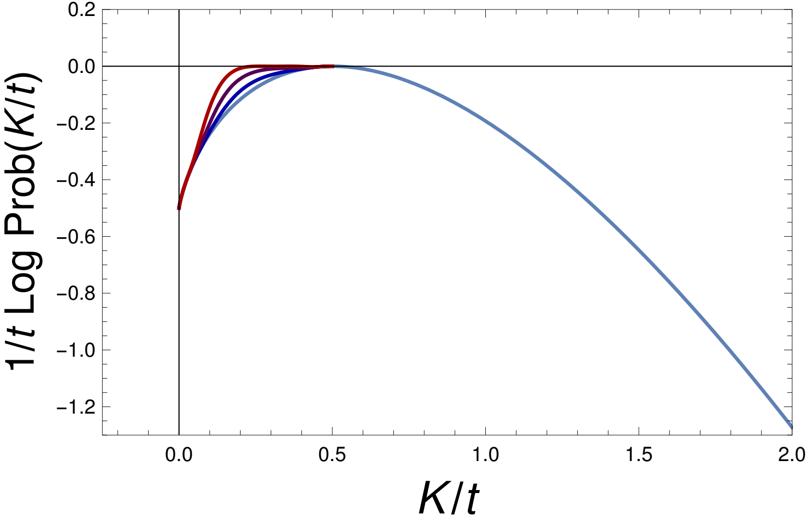
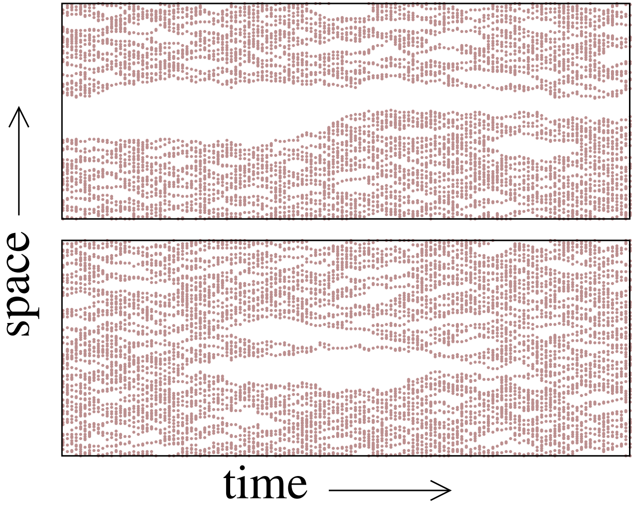
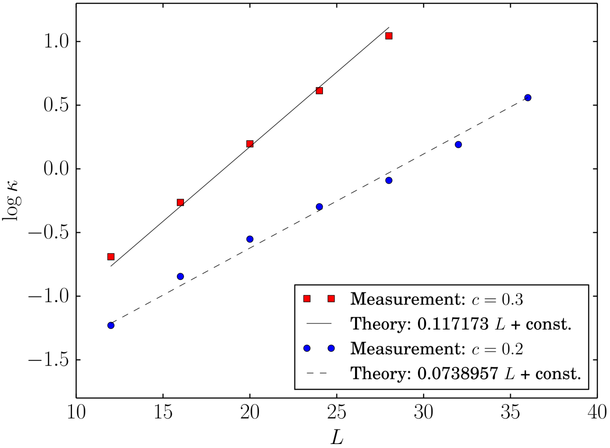


$\mathrm{Prob}[K] \sim e^{t\,\varphi(K/t)}$ Finite-time & -size scalings matter.

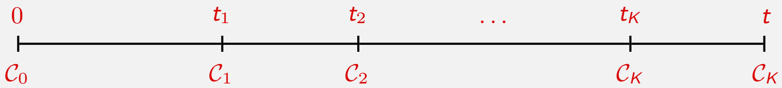


[Merolle, Garrahan and Chandler, 2005]



Exponential divergence of the susceptibility

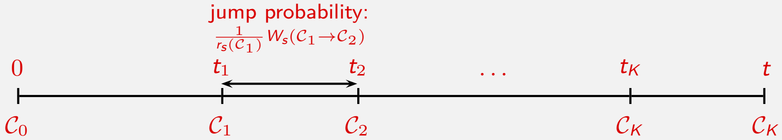
Explicit construction (1/3)



Probability-preserving contribution

$$\partial_t \hat{P}(\mathcal{C},t) = \sum_{\mathcal{C}'} \Big\{ \underbrace{W_s(\mathcal{C}' \to \mathcal{C})\hat{P}(\mathcal{C}',t)}_{\text{gain term}} - \underbrace{W_s(\mathcal{C} \to \mathcal{C}')\hat{P}(\mathcal{C},t)}_{\text{loss term}} \Big\}$$

Explicit construction (1/3)



Which configurations will be visited?

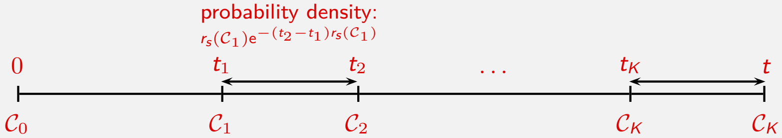
Configurational part of the trajectory: $\mathcal{C}_0 \to \ldots \to \mathcal{C}_K$

$$\text{Prob}\{\text{hist}\} = \prod_{n=0}^{K-1} \frac{W_s(\mathcal{C}_n \to \mathcal{C}_{n+1})}{r_s(\mathcal{C}_n)}$$

where

$$r_s(\mathcal{C}) = \sum_{\mathcal{C}'} W_s(\mathcal{C} \to \mathcal{C}')$$

Explicit construction (2/3)



When shall the system jump from one configuration to the next one?

- probability density for the time interval $t_n - t_{n-1}$

$$r_s(\mathcal{C}_{n-1})\mathrm{e}^{-(t_n-t_{n-1})r_s(\mathcal{C}_{n-1})}$$

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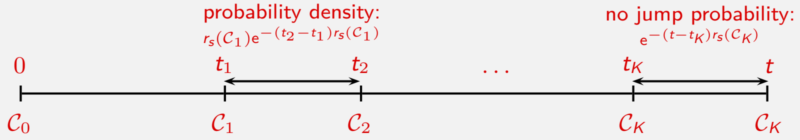

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- probability not to leave $\mathcal{C}_K$ during the time interval $t - t_K$

$$\mathrm{e}^{-(t-t_K)r_s(\mathcal{C}_K)}$$

Explicit construction (3/3)

$$\partial_t \hat{P}(\mathcal{C},s) = \underbrace{\sum_{\mathcal{C}'} W_s(\mathcal{C}' \to \mathcal{C})\hat{P}(\mathcal{C}',s) - r_s(\mathcal{C})\hat{P}(\mathcal{C},s)}_{\text{modified dynamics}} \;+\; \underbrace{\delta r_s(\mathcal{C})\hat{P}(\mathcal{C},s)}_{\text{cloning term}}$$

How to take into account loss/gain of probability?

- handle a large number of copies of the system
- implement a selection rule: on a time interval Δt a copy in config $\mathcal{C}$ is replaced by $e^{\Delta t\, \delta r_s(\mathcal{C})}$ copies
- $\psi(s)$ = the rate of exponential growth/decay of the total population
- optionally: keep population constant by non-biased pruning/cloning

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Biological interpretation

- copy in configuration $\mathcal{C} \equiv$ organism of **genome** $\mathcal{C}$
- dynamics of rates $W_s \equiv$ **mutations**
- cloning at rates $\delta r_s \equiv$ **selection** rendering atypical histories typical