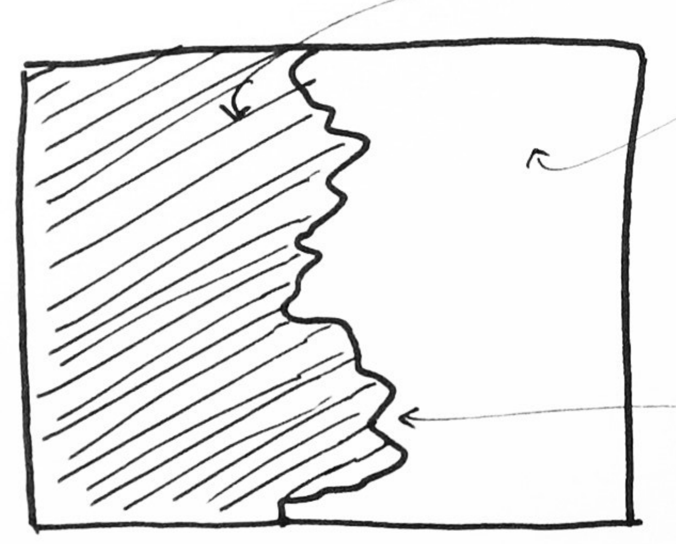
Séminaire @ GdT "Processus Stochastiques, systèmes hors d'équilibre"
7 février 2013

Statique & Dynamique d'Interfaces dans un milieu corrélé : ce que l'on apprend du polymère dirigé

[Statics & Dynamics of Interface in a correlated medium what we learn from the Directed Polymer]

References : E. Agoritsas, V.L., T. Giamarchi { arXiv: 1209.0567 (2012) ; Physica B 407 1725 (2012) }
S. Bustingorry, G. Schehr, EA, VL, TG : PRE 86 031144 (2012)

1. Physical Problem.



Two different phases
E.g.: . ferromagnetic ↑ and ↓ phases
. liquid & solid
. propagation fronts

Fluctuating $\boxed{\text{interface}}$

Questions :

→ What are the fluctuations ?

Due to { . temperature ; . quenched disorder }
[inhomogeneities of the medium]

× of the interface geometry

× of the interface distribution

→ What is the response to a driving field ?

× mean velocity

× changes in geometry

② Model : Directed Polymer in a Random Medium

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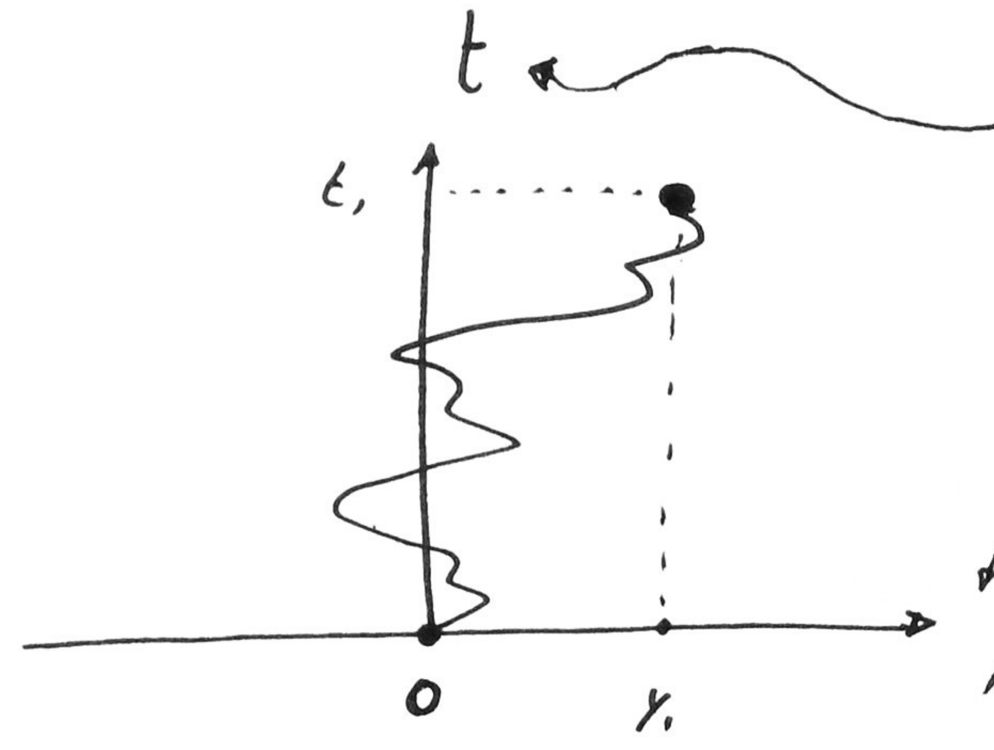


t: { spatial longitudinal coordinate of the interface
≡ time for the growth of a directed polymer (DP)

y: { spatial transverse coordinate of the interface & the DP

- Energy of a 'string' $y(t)_{0\le t\le t_1}$ in a quenched potential $V(t,y)$

$$\mathcal{H}_V[y(t);t_1] = \int_0^{t_1} dt \left\{ \underbrace{\frac{1}{2} c\, (\partial_t y)^2}_{} + \underbrace{V\big(t, y(t)\big)}_{} \right\}$$

Elastic contribution. Tends to <u>flatten</u>

Disorder contribution. Tends to <u>distort</u>.

- Partition function : weight of trajectories $(0,0) \rightsquigarrow (t_1, y_1)$

$$\boxed{Z_V(t_1, y_1) = \int_{y(0)=0}^{y(t_1)=y_1} \mathcal{D}y(t)\; e^{-\beta \mathcal{H}_V[y(t);t_1]}}$$

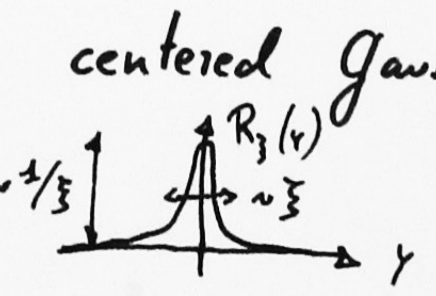
$\beta = \frac{1}{T}$ = inverse temperature

⚠ Normalization $$Z_{V\equiv 0}(t_1, y_1) = \frac{e^{-\frac{1}{2}\frac{y_1^2}{Tt/c}}}{\sqrt{2\pi \frac{Tt}{c}}}$$ ("Brownian" or "termal" propagator)

Technically speaking : the path integral $\int \mathcal{D}y(t)$ is a Wiener measure.

- Disordered Potential : $V(t,y)$ has a centered gaussian distribution

$$\overline{V(t,y)\,V(t',y')} = D\, \delta(t'-t)\, R_\xi(y'-y)$$



$\int dy\, R_\xi(y) = 1$

D: intensity of disorder

R_ξ: correlator of the disorder

Scaling : $\boxed{R_\xi(y) = \frac{1}{\xi} R_1(y/\xi)}$

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③ Equations of Evolution ; Symmetries

- Stochastic Heat Equation for the partition function :

$$\begin{cases} \partial_t Z_V(t,y) = \frac{T}{2c}\,\partial_y^2 Z_V - \frac{1}{T} V(t,y)\, Z_V(t,y) \\ Z_V(0,y) = \delta(y) \quad \text{(initial condition)} \end{cases}$$

(Feynman-Kac Formula)

⚠ "Multiplicative noise"

- Kardar - Parisi - Zhang equation for the Free Energy : $F_V(t,y)$

KPZ

$$Z_V(t,y) = e^{-\frac{1}{T} F_V(t,y)}$$

$$\boxed{\partial_t F_V(t,y) = \frac{T}{2c}\,\partial_y^2 F_V - \frac{1}{2c}\left(\partial_y F_V\right)^2 + V(t,y)}$$

× ⚠ Additive noise, at the price of a non-linear term $(\partial_y F_V)^2$
Initial condition : ill-defined ("sharp wedge")

× Related to : a zoology of problems

Population dynamics — Burgers hydrodynamics — Quantum Bose gases — last-passage percolation ...

× Caveat : highly non trivial mathematical meaning / construction ⚠

- Statistical Tilt Symmetry :

Define $\bar{Z}_V(t,y) = Z_V(t,y) / Z_{V=0}(t,y)$ 'reduced partition function'

or equiv^{lly} $\bar{F}_V(t,y) = F_V(t,y) - \underbrace{\left\{ c\frac{y^2}{2t} + \frac{T}{2}\log\frac{2\pi T t}{c} \right\}}_{F_{th}(t,y)}$

Then $\boxed{\bar{F}_V(t,y) \text{ is invariant by translation along } y \text{, in distribution}}$

Equation of evolution : tilted KPZ evolution

$$\partial_t \bar{F}_V(t,y) + \frac{y}{t}\partial_y \bar{F}_V(t,y) = \frac{T}{2c}\partial_y^2 \bar{F}_V - \frac{1}{2c}\left(\partial_y \bar{F}_V\right)^2 + V(t,y)$$

$\bar{F}_V(0,y) = 0$ — Simple initial condition

4. KNOWN RESULTS @ $\xi = 0$ (NO DISORDER CORRELATIONS)

Review of existing results ; our contribution is @ $\xi > 0$.

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- **IN THE STEADY-STATE** $(t \to \infty)$.

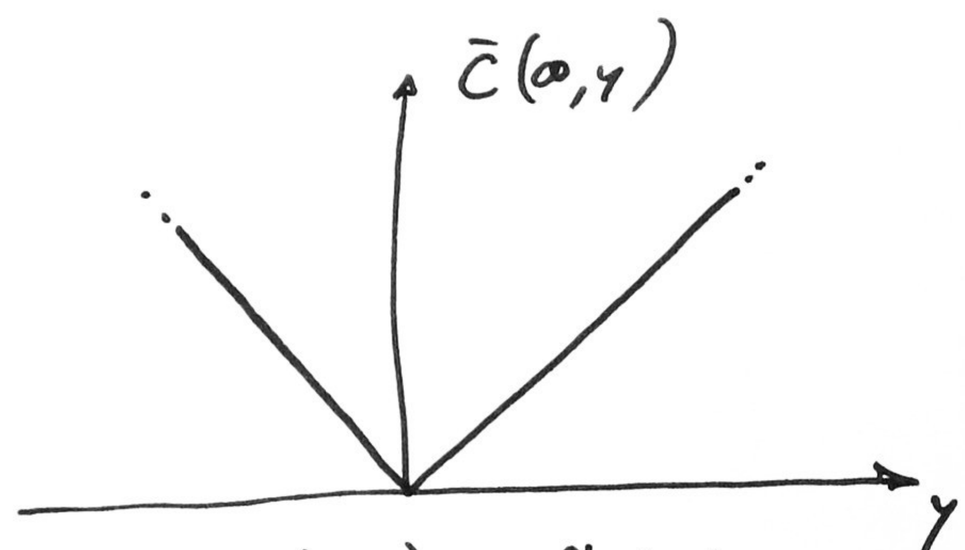
$$\text{Prob}\,[F_V] \propto e^{-\frac{1}{2}\frac{1}{\tilde{D}}\int dt\,(\partial_y F)^2}\,, \quad \boxed{\tilde{D} = \frac{cD}{T}}$$ [Not valid anymore @ $\xi > 0$]

In other words, F_V is a Brownian motion (of coordinate y) in the steady state

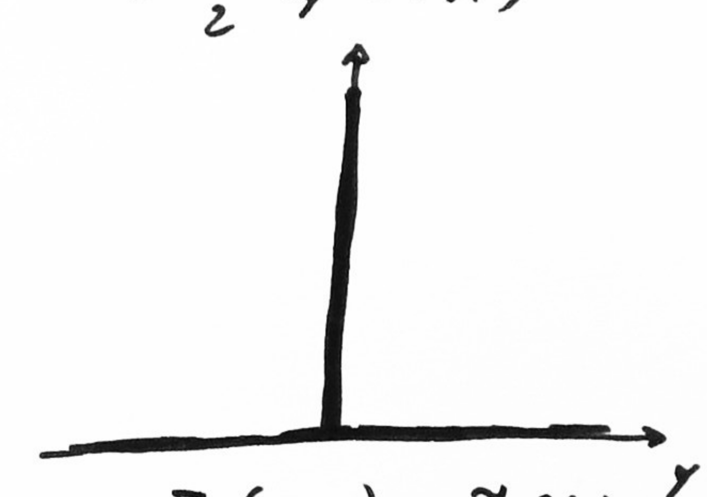
⚠ SAME RESULT IN THE ABSENCE OF THE NONLINEARITY

Convenient representation : consider the following 2-point correlation funct°

$$\bar{C}(t,y) = \overline{\left(\bar{F}_V(t,y) - \bar{F}_V(t,0)\right)^2} \qquad \bar{R}(t,y) = \overline{\partial_y \bar{F}_V(t,y)\,\partial_y \bar{F}_V(t,0)} = \frac{1}{2}\partial_y^2 \bar{C}(t,y)$$



$\Leftrightarrow$



$$\bar{C}(\infty,y) = \tilde{D}\,|y| \qquad \bar{R}(\infty,y) = \tilde{D}\,\delta(y)$$

- **INTERLUDE : DERIVATION OF 2/3 EXPONENT (INFORMALLY) :**

If, at large times, $\bar{F}(t,y)$ still has a distribution close to gaussian,

$$F_{th}(t,y) + \bar{F}(t,y) \overset{d}{=} \underbrace{\frac{a^2}{b}\frac{c\bar{y}^2}{2\bar{t}} + a^{\frac{1}{2}}\tilde{D}^{\frac{1}{2}}\,\bar{F}_1(\bar{t},\bar{y})}_{\text{Chose } a = \left(\frac{\tilde{D}}{c^2}\right)^{\frac{1}{3}} b^{\frac{2}{3}} \text{ for those two terms to scale in the same way}} \quad \text{upon } \begin{array}{l} y = a\bar{y} \\ t = b\bar{t} \end{array}$$

($\to \tilde{D} = 1$)

Then, introducing the roughness $B(t) = \overline{\langle y(t)^2 \rangle} = \overline{\int dy\, y^2 Z_V(t,y) \Big/ \int dy\, Z_V(t,y)}$ one has chosing $b = t$

$$B(t) = \left[\frac{\tilde{D}}{c^2}\right]^{\frac{2}{3}} t^{\frac{4}{3}}\; \overline{\frac{\int d\bar{y}\;\bar{y}^2 \exp\left\{-\frac{1}{T}\left[\frac{\tilde{D}^2}{c}t\right]^{\frac{1}{3}}\left[\frac{\bar{y}^2}{2} + \bar{F}_1(1,\bar{y})\right]\right\}}{\int d\bar{y}\; \exp\{\quad (\text{idem}) \quad\}}}$$

Saddle in the large-time limit : reached for the same t-independent $\bar{y}$

$\bar{y}^\star$ is the $\bar{y}$ at saddle

$$\boxed{B(t) = \left[\frac{\tilde{D}}{c^2}\right]^{\frac{2}{3}} t^{\frac{4}{3}}\; \underbrace{\overline{\bar{y}^{\star 2}}}_{\text{independent of } t}}$$

$\boxed{\frac{2}{3} \text{ exponent}}$

↳ The numerator & denominator compensate

- RECENT RESULTS : UNIVERSAL KPZ FLUCTUATIONS

Dotsenko
Amir Corwin Quastel
Sasamoto - Spohn
Calabrese - Le Doussal - Rosso

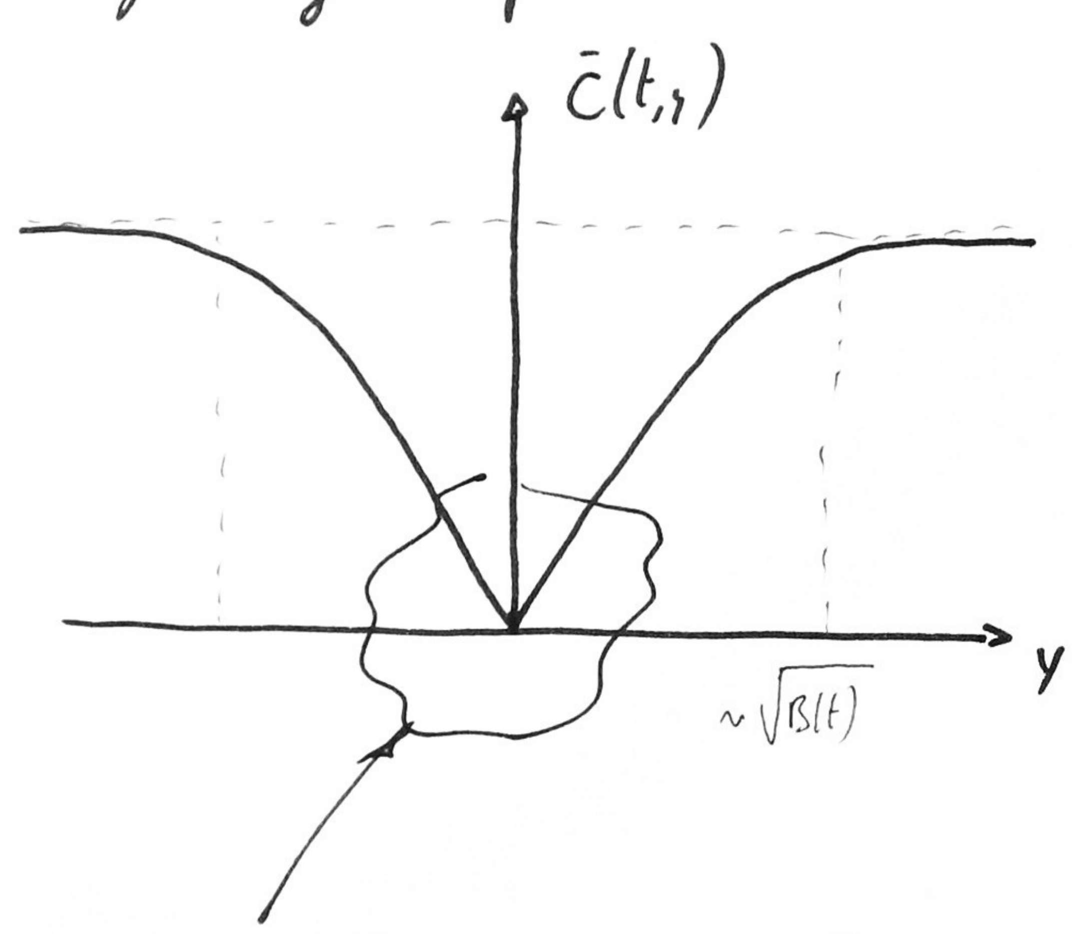
In the large t limit
(up to numerical constants)

$$\boxed{\bar{F}_V(t,y) \overset{d}{=} \left(\tilde{D}\sqrt{B(t)}\right)^{1/2} A_2\left(y/\sqrt{B(t)}\right)}$$

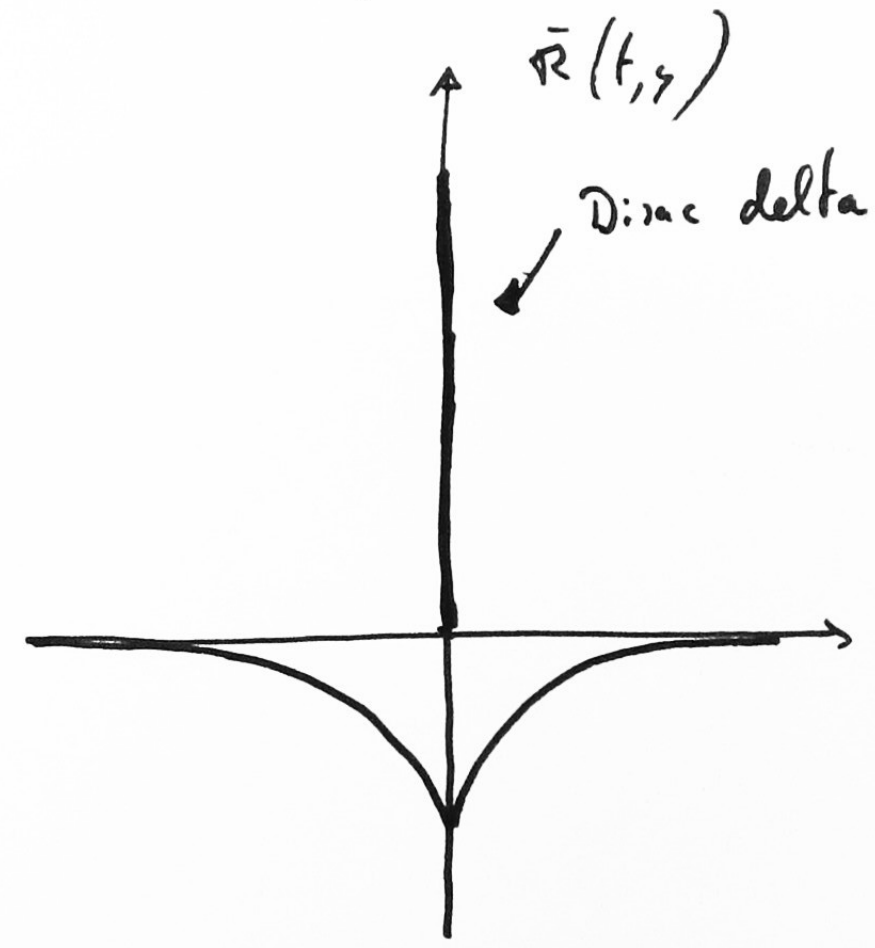
$$B(t) = \left(\frac{\tilde{D}}{c^2}\right)^{2/3} t^{4/3} \quad , \quad \boxed{\tilde{D} = \frac{cD}{T}}$$

$A_2(\bar{y})$: Airy$_2$ process universal (independent of c D T)

Corresponding 2-point correlator $\bar{C}(t,y)$ & $\bar{R}(t,y)$:



One recovers the Brownian scaling at large times



$$\bar{R}(t,y) = \tilde{D}\,\delta(y) - \underbrace{\frac{\tilde{D}}{\sqrt{B(t)}}\,\text{funct}^{\circ}\left(\frac{y}{\sqrt{B(t)}}\right)}_{\to 0 \text{ as } t\to\infty \text{ @ fixed } y}$$

$$\bar{C}(t,y) = \tilde{D}\sqrt{B(t)}\;\hat{C}\left(y/\sqrt{B(t)}\right)$$

parameter-free scaling function $\hat{C}(\bar{y})$

Rq : $\forall t<\infty$, $\int_{\mathbb{R}} \bar{R}(t,y)\,dy = 0$
but @ $t=\infty$, $\int_{\mathbb{R}} \bar{R}(t,y)\,dy = \tilde{D}$

5. Numerical Results at $\xi > 0$:

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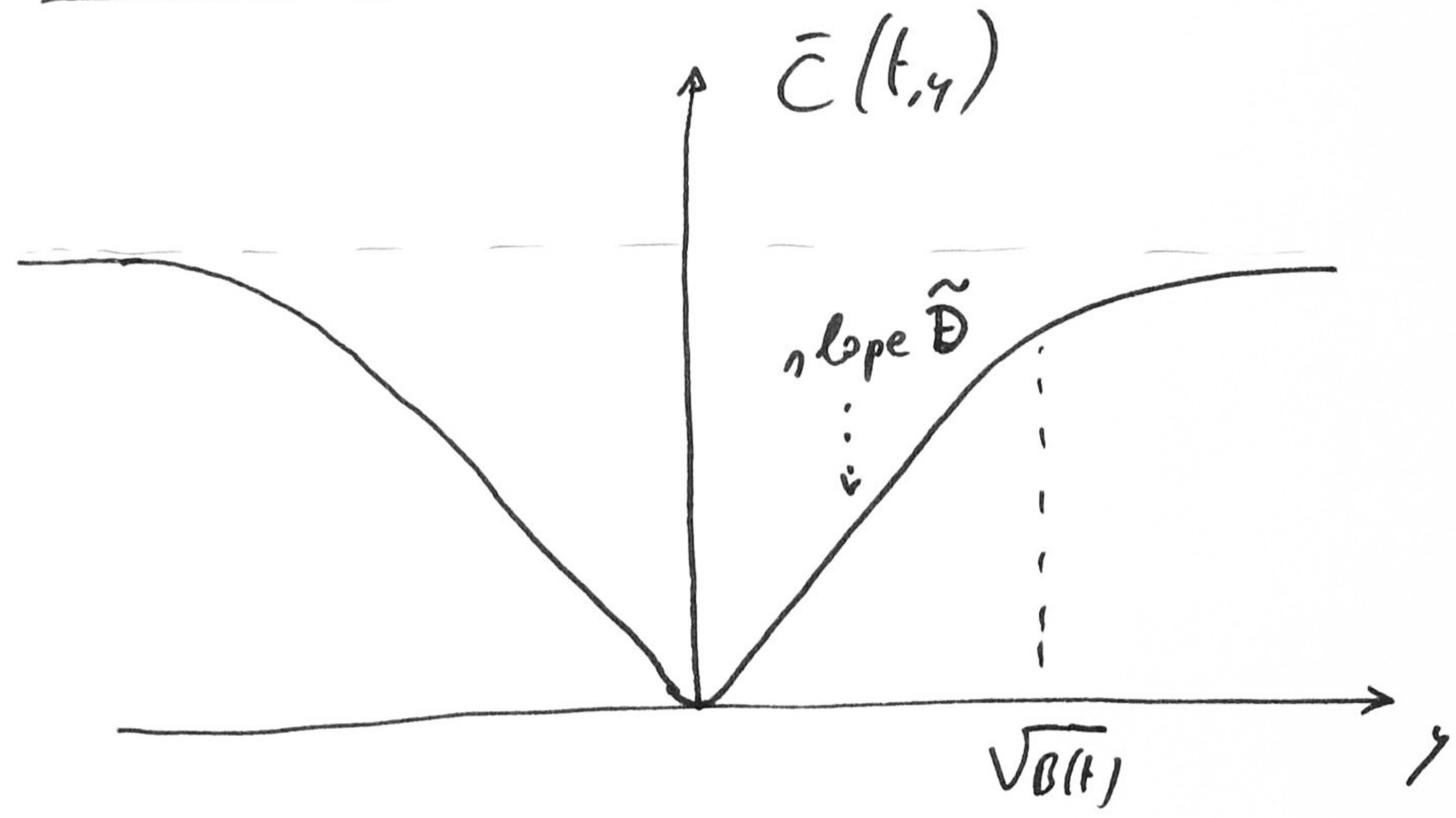
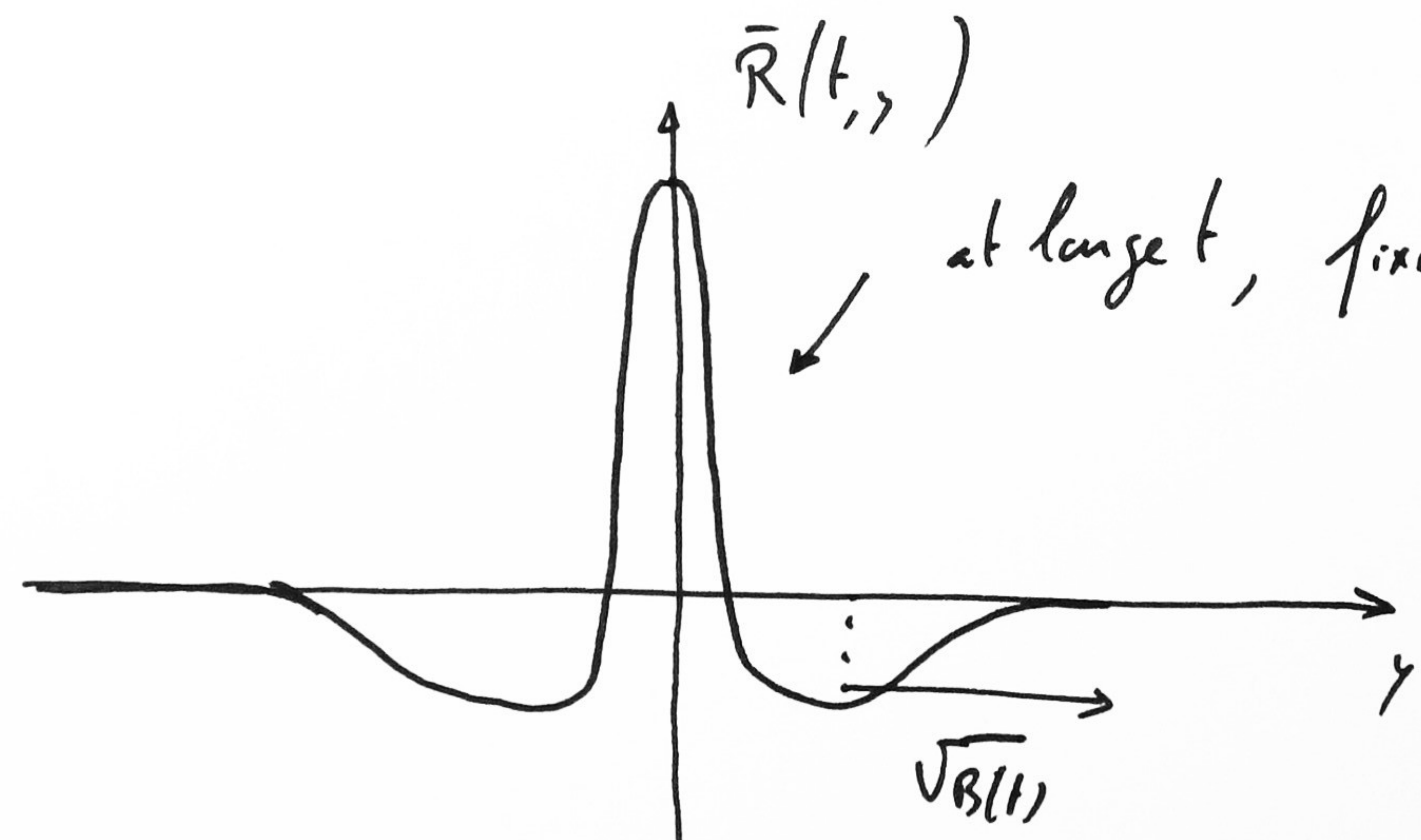


Compatible with

$$\bar{C}(t,y) = \tilde{D}\sqrt{B(t)}\ \hat{C}_{\xi/\sqrt{B(t)}}\left(y/\sqrt{B(t)}\right)$$

scaling function $\hat{C}_{\hat{\xi}}(\hat{y})$ to determine



"bumps" $\searrow 0$ as $t \to \infty$

ensure the exact conserved quantity

$$\int_{\mathbb{R}} dy\ \bar{R}(t,y) = 0 \quad \forall t < \infty .$$

Simplest questions : what is the stationary $\bar{R}(\infty,y)$ at $\xi > 0$?

what is the height $\bar{R}(\infty,0)$ of the peak ?

$\hookrightarrow$ no analytic answer

6. Analytical Results @ $\xi > 0$: Scaling

- <u>Why is it interesting to focus on $\tilde{D}$?</u> [$\tilde{D}$ ≡ amplitude of free-energy fluctuations]

even at $\xi > 0$, if the scaling argument at large t holds, one has

$$B(t) \underset{t\to\infty}{\sim} \underbrace{\left(\frac{\tilde{D}}{c^2}\right)^{\frac{2}{3}}} t^{4/3} \times \left(\begin{array}{l}\text{numerical constant}\\ \text{independent on } c\,D\,T\,\xi\end{array}\right)$$

if indeed $\tilde{D}$ depends on ξ,
it means that ξ can have an influence even at scales much larger than ξ $(t\to\infty)$

<u>Rq</u>: $B(t)$ is a quantity that can be measured experimentally.

- <u>Rescaling for the roughness, from its definition</u>:

a. $B(t; c\,D\,T\,\xi) = \left(\frac{T^3}{cD}\right)^2 B\left(\frac{t}{\frac{T^5}{cD^2}}; 1, 1, 1, \underbrace{\frac{\xi}{\frac{T^3}{cD}}}\right)$ (exact)

Defining $\boxed{T_c := (\xi c D)^{1/3}}$, ≈ 0 if $T \gg T_c$

Hence, in the high-temperature limit $(T \gg T_c)$:

$$B(t); c\,D\,T\,\xi) = \left(\frac{T^3}{cD}\right)^2 B\left(\frac{t}{T^5/cD^2}; 1\ 1\ 1\ 0\right) \sim \left(\frac{\tilde{D}}{c^2}\right)^{2/3} t^{4/3}, \quad \boxed{\tilde{D} = \frac{cD}{T}}$$

b. $B(t; c\,D\,T\,\xi) = \xi^2\, B\left(\frac{t}{\frac{T_c^5}{cD^2}}; 1\ 1\ \frac{T}{T_c}\ 1\right)$

Hence: if the $T\to 0$ limit is well behaved in this expression

one recovers now $B(t) \sim \left(\frac{\tilde{D}}{c^2}\right)^{2/3} t^{4/3}$ $\quad \boxed{\tilde{D} = \frac{cD}{T_c}}$

<u>Conclusion</u>: From (advanced) scaling,

$$\boxed{\tilde{D} \sim \begin{cases} cD/T & T \gg T_c \\ cD/T_c & T \ll T_c \end{cases}}$$

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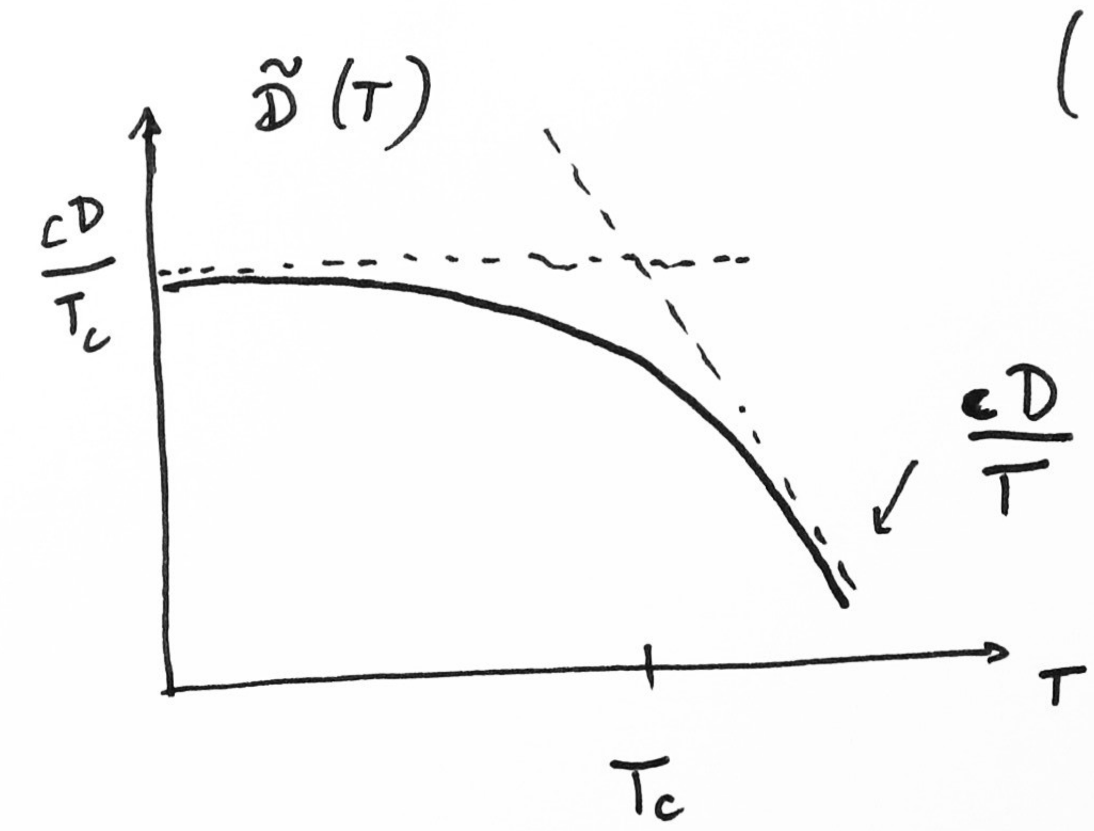
· Justification of the $T \to 0$ limit : optimal trajectories

The rescaling b page 7 writes explicitely
$(\bar{t}_1 = t_1 / T_c^3 / cD^2)$

$$\overline{B}(t_1; cDT\xi) = \xi^2 \cdot \frac{\int_{\bar{y}(0)=0} \mathcal{D}\bar{y} \; e^{-\frac{T_c}{T}\int_0^{\bar{t}_1} dt \left\{ \frac{1}{2}(\partial_t \bar{y})^2 + V_1(t,\bar{y}(t)) \right\}} \cdot y(\bar{t}_1)^2}{\int_{\bar{y}(0)=0} \mathcal{D}\bar{y} \quad (\text{idem})}$$

The $T \to 0$ limit correspond to a path-integral saddle point asymptotics.
Denoting $y^*(t)$ the optimal trajectories, numerator & denominator
(common to numerator & denominator) compensate
and the $T \searrow 0$ limit is well behaved.

· Numerical results :



(log-log plot)

works very well!

7. Analytical Results @ $\xi > 0$: Correlation Functions

Interpret KPZ equation as a "Langevin" equation $\to$ use Itō formula

$$\partial_t \bar{R}(t,y) + \frac{1}{t}[\text{tilt}] = \frac{T}{c}\partial_y^2 \bar{R} - \frac{1}{c}\bar{R}_3(t,y) - D\,R_3''(y)$$

$\leftarrow$ Itō term
R_3: 3 point function

Hypothesis : $\xi \updownarrow \tilde{D}/\xi$ for the correlator

① : @ $T=0$ one gets $\boxed{\tilde{D} = cD/T_c}$

② : @ generic T: denoting $\tilde{D} = f \cdot \frac{cD}{T}$ one gets

$$f^\gamma = \left(\frac{T}{T_c}\right)^\gamma (1-f) \qquad \gamma = 3/2$$

@ $t \to \infty$

describes the full crossover
$T \gg T_c$
$T \ll T_c$

Remarks :

- if fluctuations of $\bar{F}$ were Gaussian, $R_3 = 0$ and $\tilde{D} = \frac{cD}{T}$

  $\Rightarrow$ $\boxed{\tilde{D} \neq \frac{cD}{T} \text{ @ } \xi > 0 \text{ is a hallmark of non-gaussian fluctuations of } \bar{F}}$

- if one removes the non-linearity (Edwards-Wilkinson): $\tilde{D} = \frac{cD}{T}$
- time-dependence can be studied.
- Gaussian Variational Method yields the same equation for f with $\gamma = 6$
  $\to$ this modifies the crossover, but not the asymptotics $T \gtrless T_c$.

8. Links to Experiments

- Ferromagnetic interfaces: well-described by $y(t)$

  $\tilde{D}$ not easy to correlate to $cDT\xi$

  although: scalings in temperature are measured in Pt/Co/Pt films

- Turbulence in liquid crystal: [Takeuchi & Sano]

  → $F(t,y)$ is directly related to the height of an interface.

  Observed Airy_2 & Airy_1 scalings.

  → Measure of $\tilde{D}$ through correlat° functions:

  seems compatible with a regime where ξ matters

(Rq: in this experiment, "low temperature" ⟺ high velocity)

$$\partial_t F(t,y) = \nu \partial_y^2 F + \frac{\lambda}{2}(\partial_y F)^2 + V(t,y) + v$$

$\uparrow$ (λ) velocity parameter

$$\tilde{D} \sim \begin{cases} \lambda & \lambda \ll \lambda_c \quad \xi \text{ does not matter} \\ \lambda^{-1/3} & \lambda \gg \lambda_c \quad \xi \text{ matters} \end{cases}$$

9_ Dynamics & Creep :

- - - - ———> mean velocity v

Nonlinear response ("creep law") quasistatic regime

$$v \sim e^{-\frac{E_c}{T}\left(\frac{f_c}{f}\right)^{\mu}} \qquad \mu = 1/4$$

① **f_c directly related to crossover lenght :**

can be related to ξ

However T-dependence of numerically determined f_c not recovered.

② **Determination of $\mu = 1/4$:**

- FRG in dim $4-\varepsilon$, $\varepsilon \ll 1$ @ $\varepsilon = 3$!
- proposed approach from scaling with the DP @ $\xi > 0$

0d :

v related to equil. distrib @ $f > 0$

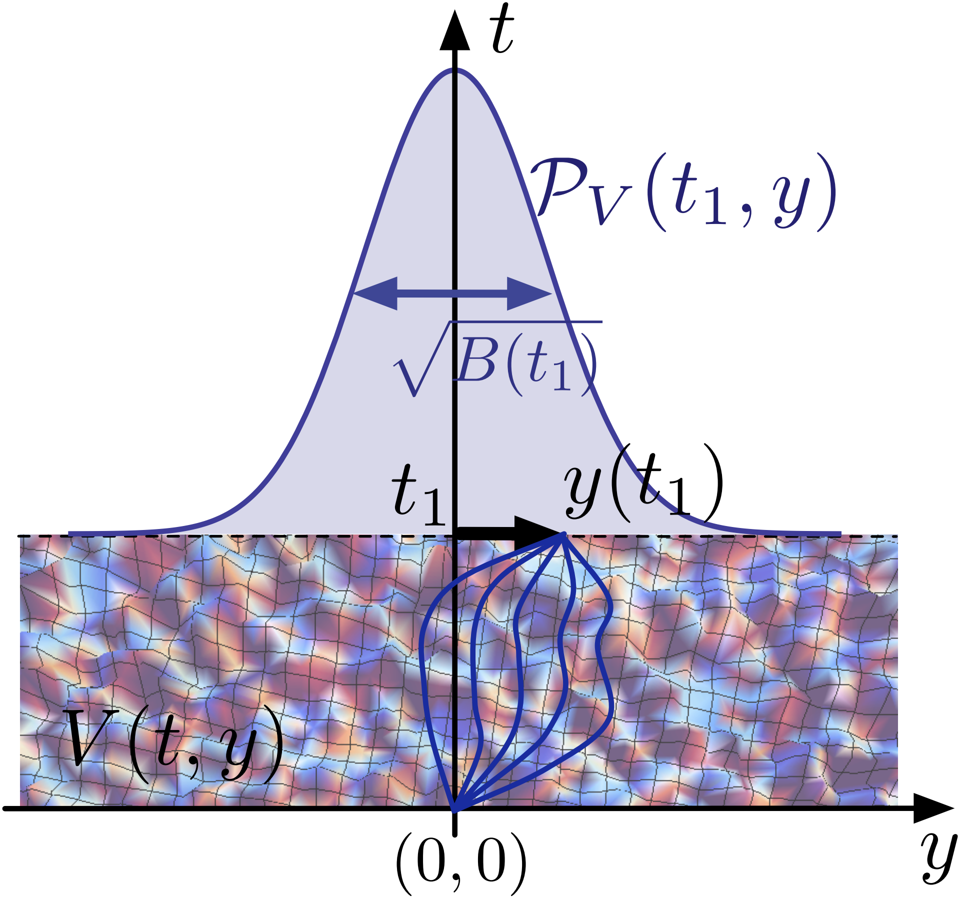
1d :

distribution at equilibrium at $f > 0$

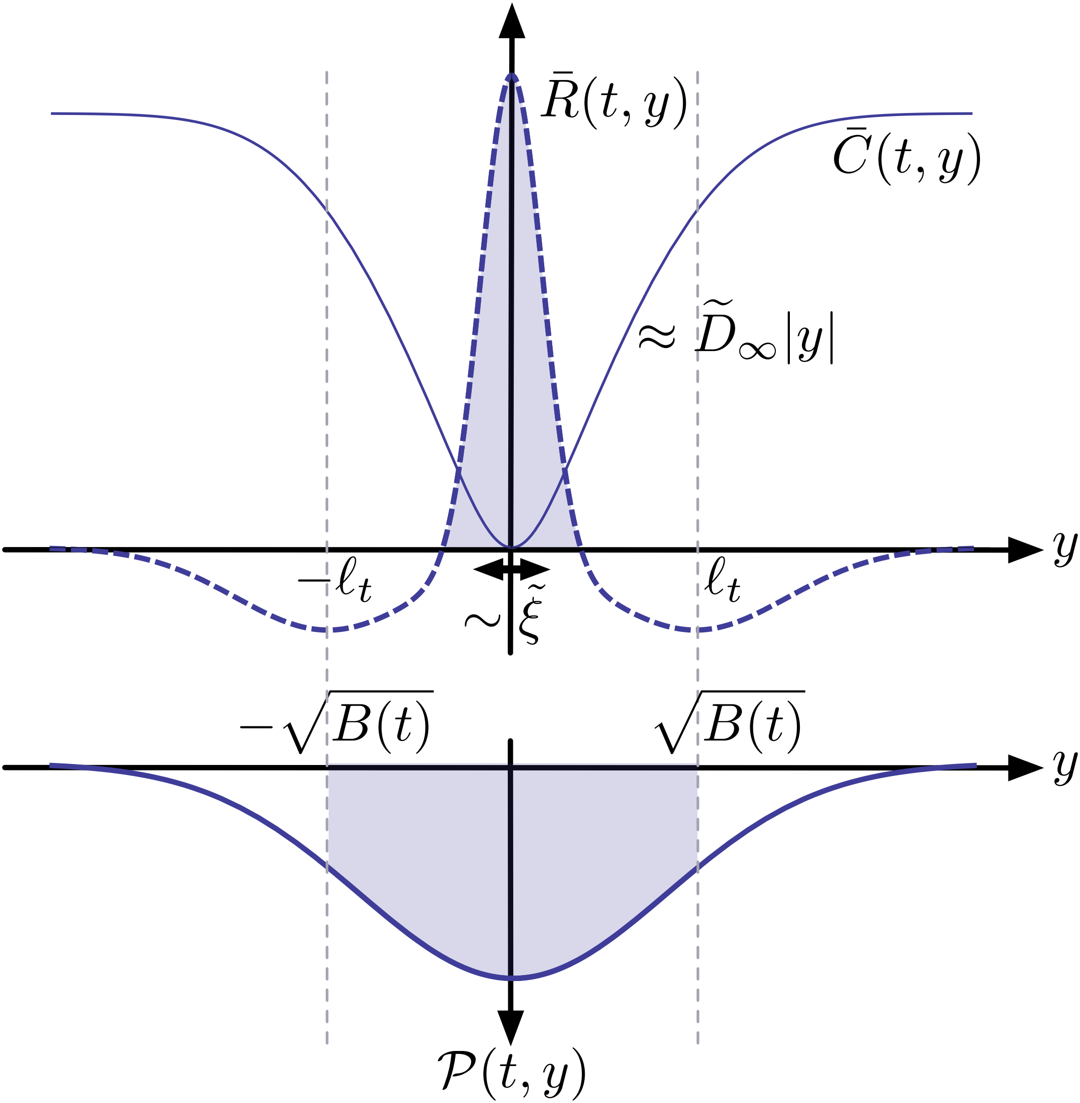
Blind translation of formula from 0d to 1d:

$\boxed{\text{yield the creep law in a large } t \text{ saddle}}$

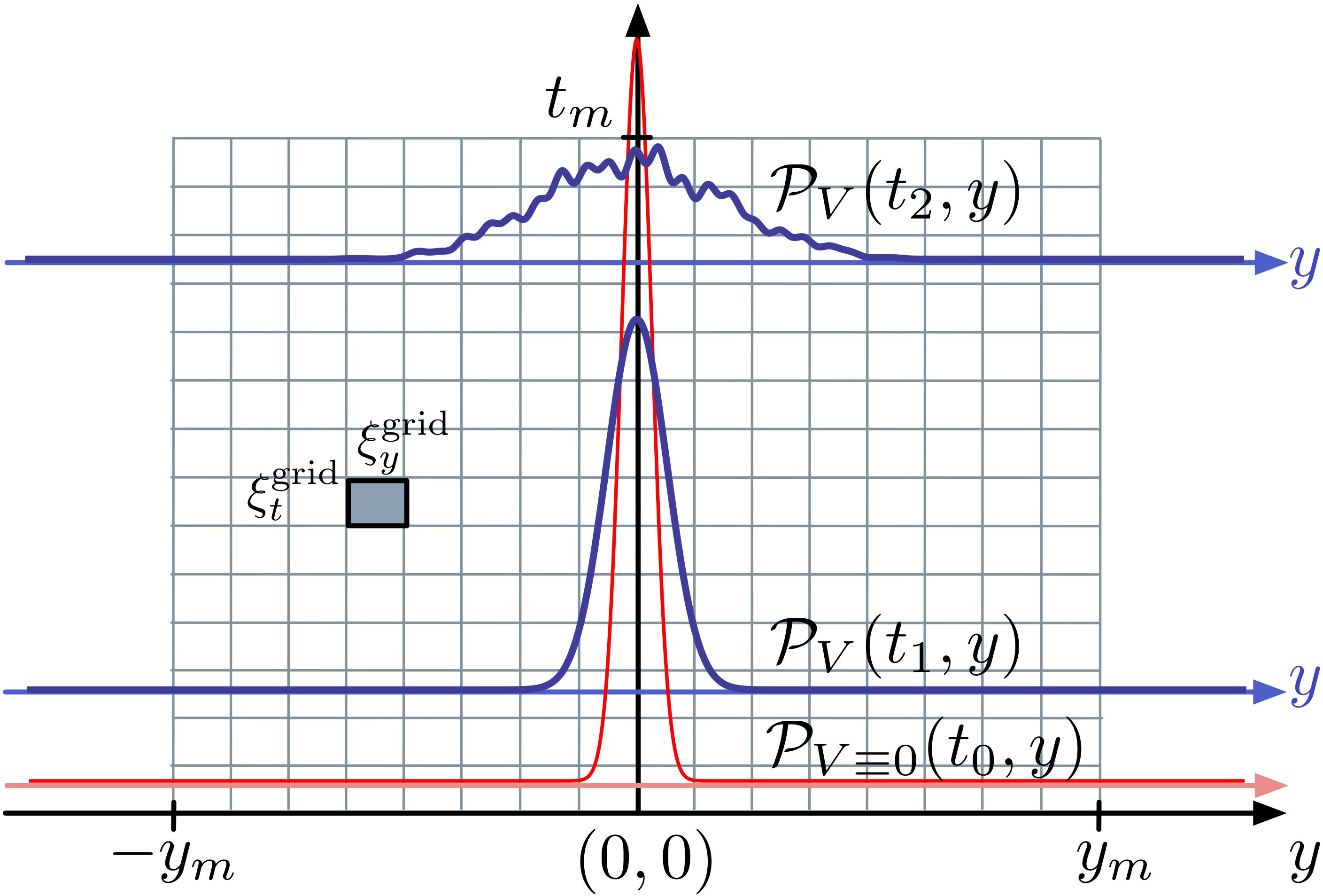
CAN THIS BE JUSTIFIED ?



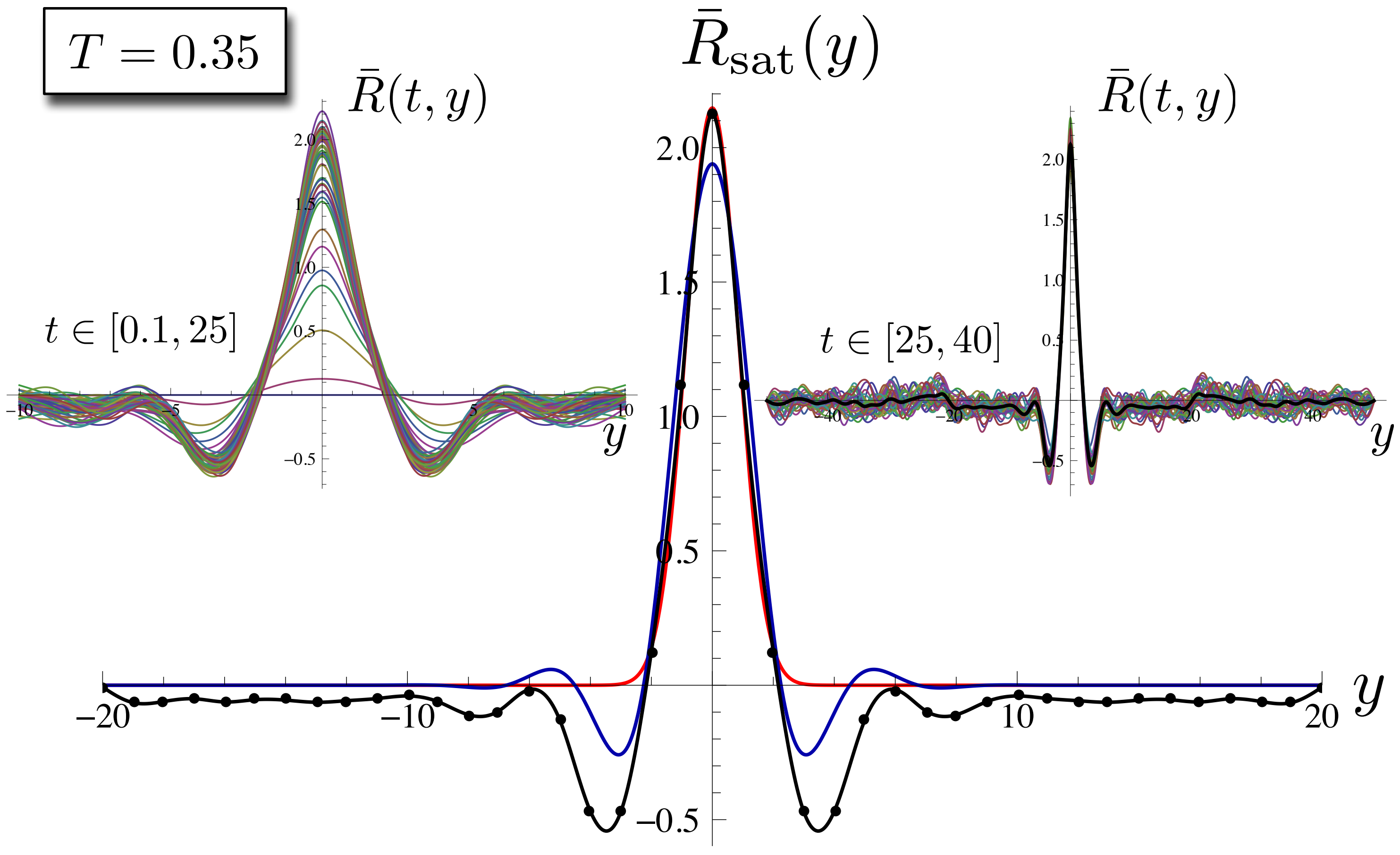
t
$\mathcal{P}_V(t_1, y)$
$\sqrt{B(t_1)}$
t_1
$y(t_1)$
$V(t, y)$
$(0, 0)$
y



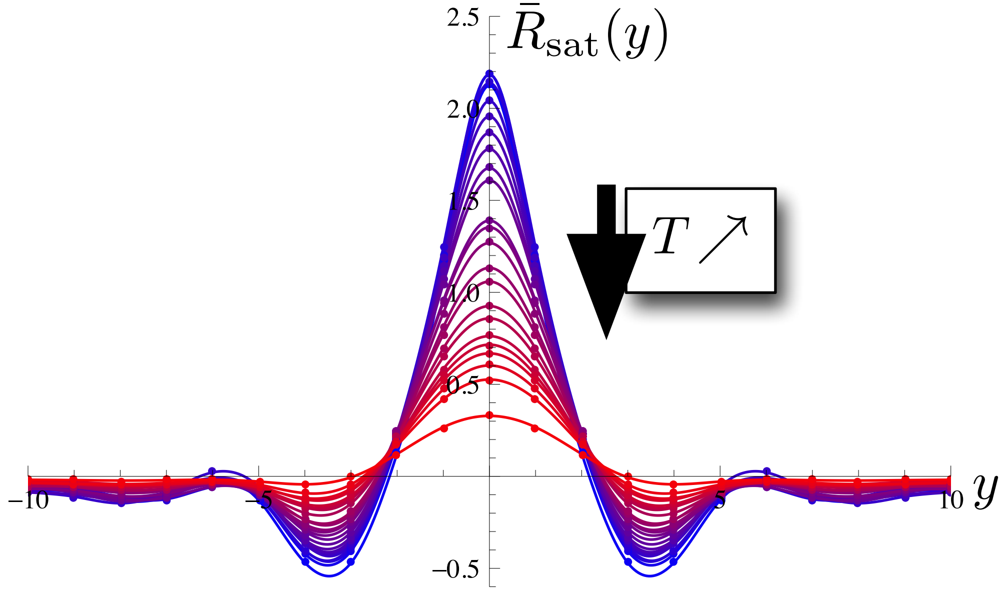
$\bar{R}(t,y)$
$\bar{C}(t,y)$
$\approx \widetilde{D}_\infty |y|$
y
$-\ell_t$
ℓ_t
$\sim \tilde{\xi}$
$-\sqrt{B(t)}$
$\sqrt{B(t)}$
y
$\mathcal{P}(t,y)$



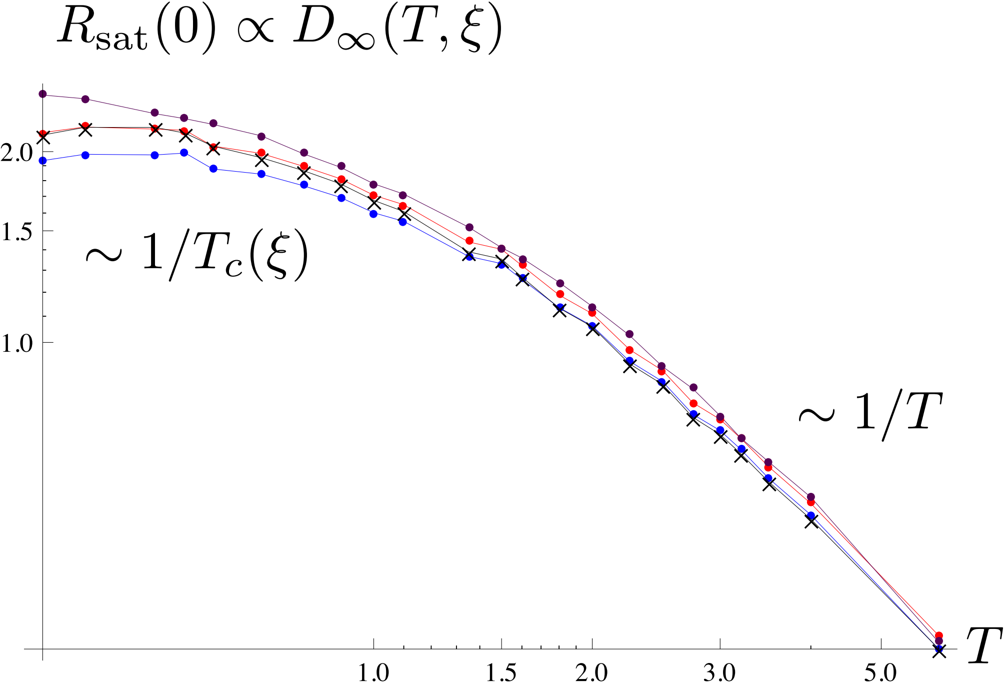
t_m
$\mathcal{P}_V(t_2, y)$
y
ξ_y^{grid}
ξ_t^{grid}
$\mathcal{P}_V(t_1, y)$
y
$\mathcal{P}_{V\equiv 0}(t_0, y)$
$-y_m$
$(0,0)$
y_m
y

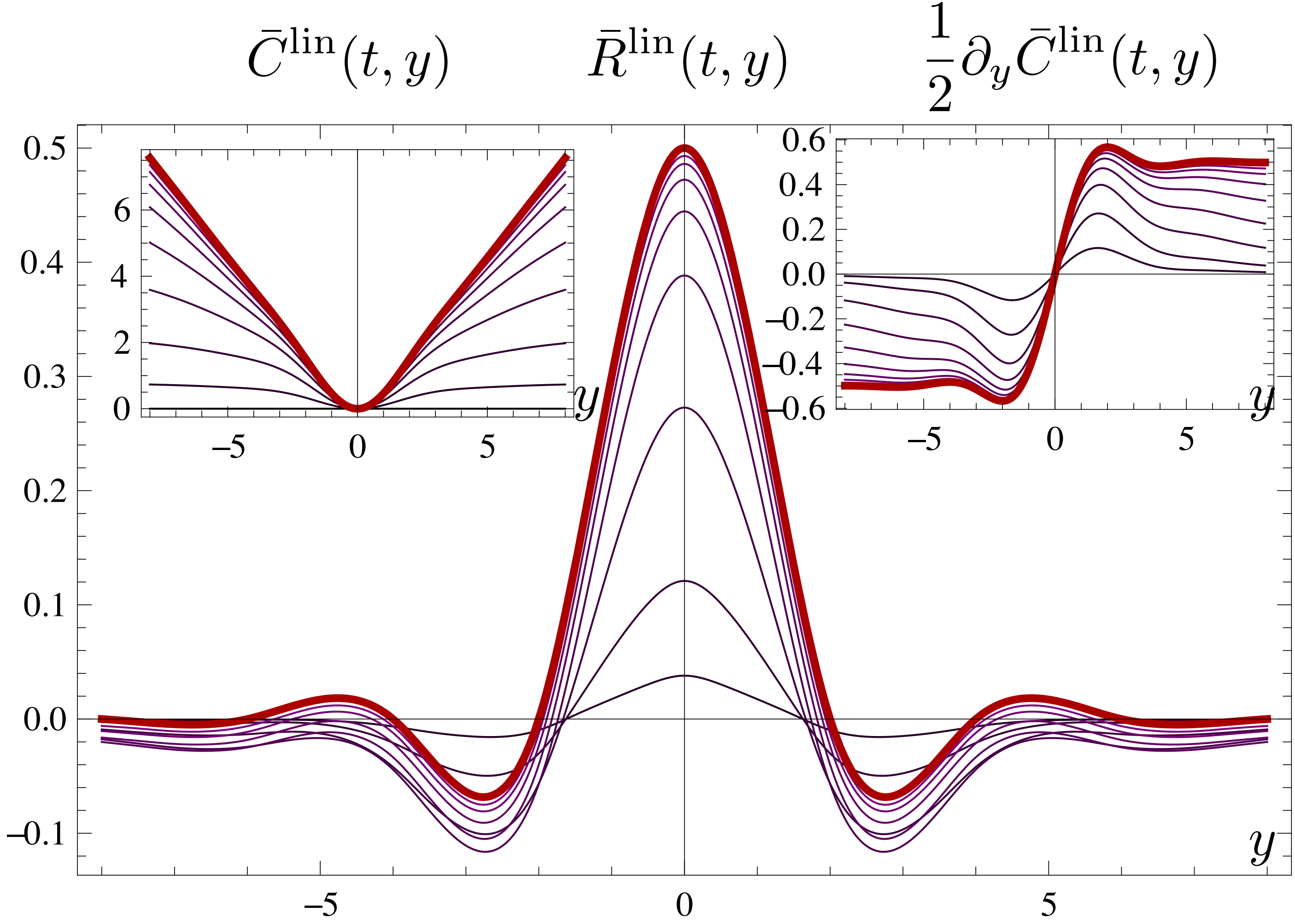


$T = 0.35$
$\bar{R}_{\rm sat}(y)$
$\bar{R}(t,y)$
$t \in [0.1, 25]$
$\bar{R}(t,y)$
$t \in [25, 40]$
y
-20
-10
10
20
2.0
1.5
1.0
0.5
-0.5



$\bar{R}_{\rm sat}(y)$
2.5
2.0
1.5
1.0
0.5
−0.5
−10
−5
5
10
y
$T \nearrow$


$\bar{R}_{\rm sat}(0) \propto \widetilde{D}_\infty(T,\xi)$
$\sim 1/T_c(\xi)$
$\sim 1/T$
T
2.0
1.5
1.0
1.0
1.5
2.0
3.0
5.0



$\bar{C}^{\mathrm{lin}}(t,y)$
$\bar{R}^{\mathrm{lin}}(t,y)$
$\frac{1}{2}\partial_y\bar{C}^{\mathrm{lin}}(t,y)$
0.5
0.4
0.3
0.2
0.1
0.0
−0.1
−5
0
5
y
6
4
2
0
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6