Finite-time implications of dynamical phase transitions in exclusion processes

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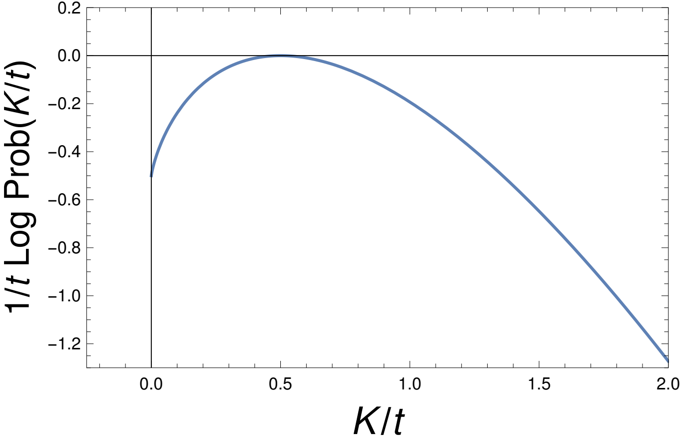
Dynamical phase transitions and anomalous fluctuations



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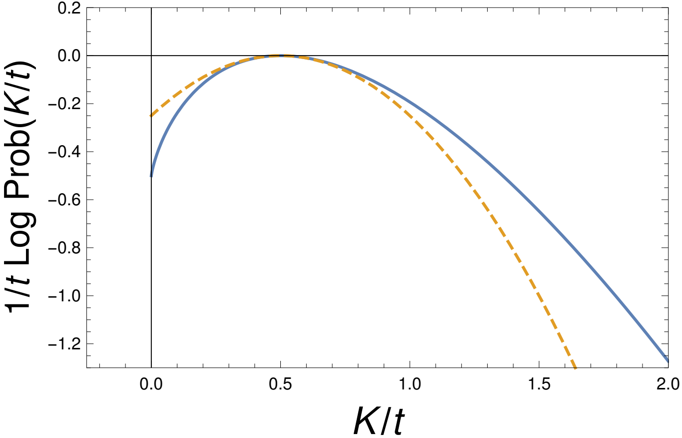
Dynamical phase transitions and anomalous fluctuations



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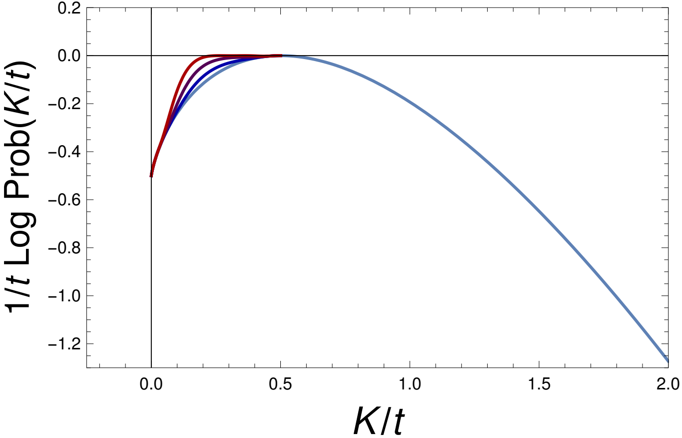
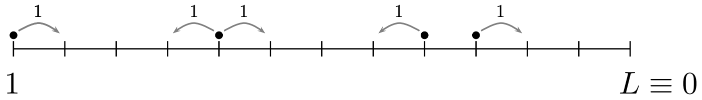


For a "time-extensive" order parameter K: $\text{Prob}[K] \sim e^{t\,\mathcal{I}(K/t)}$

Finite-time & -size scalings matter.

Large deviations in periodic space

Simple exclusion process (SSEP): L sites, periodic (or isolated)
Fixed total particle number N_0 density: $\rho_0 = N_0/L$



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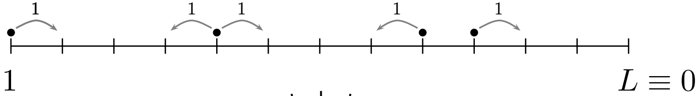

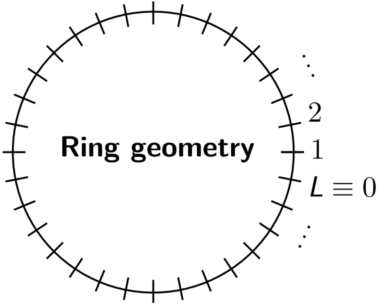

Large deviations in periodic space $\quad s \leftrightarrow$ activity K

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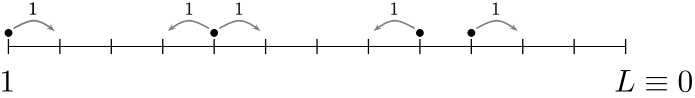

Large deviation function of "additive" observables A

$$\langle e^{-sA} \rangle = e^{\Psi(s,t)} \qquad (\Leftrightarrow \text{determining } \mathbb{P}(A,t))$$

A = total current Q on time window $[0,t]$ $= \#\overrightarrow{\text{jumps}} - \overleftarrow{\text{jumps}}$

A = total activity K on time window $[0,t]$ $= \#\overrightarrow{\text{jumps}} + \overleftarrow{\text{jumps}}$

Exact results: Lebowitz&Spohn, Derrida, Mallick, Lazarescu, Prolhac, Evans, Schutz, Popkov, ...

Macroscopic: Jona-Lasinio *et al.*, Bodineau&Derrida, Meerson *et al.* ...

Numerics: Hurtado *et al.*, Vanderzande *et al.* ...

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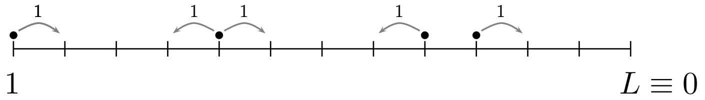

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- ⋆ Usual settings, *at large time* $t \to \infty$: $\Psi(s,t) \sim t\,\psi(s)$

$$\lim_{t\to\infty} \frac{1}{t} \langle A^k \rangle_c = (-1)^k \psi^{(k)}(0)$$

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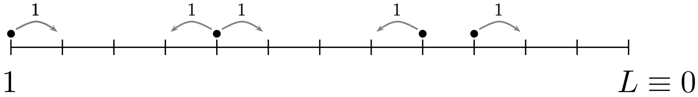

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- ⋆ Here, *at finite time*:

$$\langle A^k \rangle_c = (-1)^k \, \partial_s^k \Psi(s,t)|_{s=0}$$

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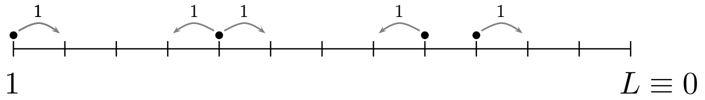

Operator representation: $\langle e^{-sA}\rangle = e^{\Psi(s,t)} = \langle -|e^{t\mathbb{W}_s}|P_\mathrm{i}\rangle$

$$\mathbb{W}_s = \sum_{k=1}^{L-1}\Big[e^{-s}\big(\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+\big) - \hat{n}_k(1-\hat{n}_{k+1}) - (1-\hat{n}_k)\hat{n}_{k+1}\Big]$$

$$= \frac{L-1}{2} - \frac{e^{-s}}{2}\mathbb{H}_\Delta^{\mathrm{XXZ}}$$

$$\mathbb{H}_\Delta^{\mathrm{XXZ}} = -\sum_{k=1}^{L-1}\big[\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z\big] \qquad \text{with} \quad \boxed{\Delta = e^s}$$

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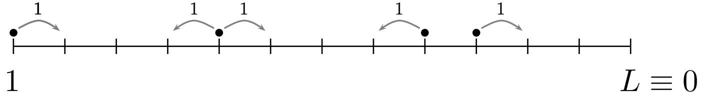

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$$= \frac{L-1}{2} - \frac{e^{-s}}{2} \mathbb{H}_\Delta^{\mathrm{XXZ}}$$

Difficulty: $s = \mathcal{O}(L^{-2})$

$$\mathbb{H}_\Delta^{\mathrm{XXZ}} = - \sum_{k=1}^{L-1} \big[\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \big]$$

with $\boxed{\Delta = e^s}$

Infinite-time results

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

From Bethe Ansatz (at large L, *keeping discreteness of Bethe roots*)

- large deviation function

$$\psi(s) = \underbrace{-2L\rho_0(1-\rho_0)s}_{\text{minimal order}} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} + \dots \quad \text{with} \quad u = L^2\rho_0(1-\rho_0)s$$

- **universal function** (singular in $u = \frac{\pi^2}{2}$)

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

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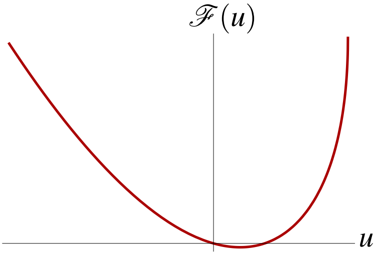

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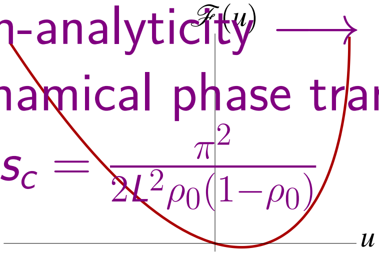


non-analyticity $\longrightarrow$ dynamical phase transition at $s_c = \dfrac{\pi^2}{2L^2\rho_0(1-\rho_0)}$

Macroscopic FT [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Saddle point evaluation at large L

$$\langle e^{-sK}\rangle = \int \mathcal{D}\rho\mathcal{D}\hat\rho\ \exp\{-L\,\mathcal{S}_s[\hat\rho,\rho]\}$$

At infinite time:

Large deviation function [assuming **uniform** profile $\rho(x)=\rho$]

$$\psi(s) = \underbrace{-s\frac{\langle K\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\int \text{ of quadratic fluctuations}} \quad \text{with} \quad u = L^2 s\frac{\sigma(\rho_0)\sigma''(\rho_0)}{8D^2}$$

Generic MFT settings, with, for the SSEP

$$\sigma(\rho) = 2\rho(1-\rho) \quad \text{and} \quad D(\rho)=1$$

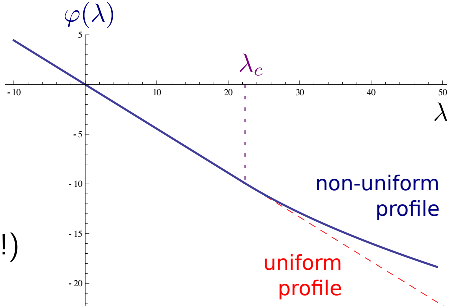
Dynamical phase transition [VL, Garrahan, van Wijland, JPA **45** 175001]

Rescaling of the large deviation function [singularity at $\lambda_c > 0$ as $L \to \infty$]

$$\varphi(\lambda) = \lim_{L\to\infty} \varphi_L(\lambda) \quad \text{with} \quad \varphi_L(\lambda) \equiv L\psi(\lambda/L^2) \qquad (\textit{i.e.: } s = \lambda/L^2)$$

Using the correct *non-uniform* saddle-point profile for $\lambda > \lambda_c$

$\lambda_c = \frac{\pi^2}{\sigma(\rho_0)}$
(can be large!)



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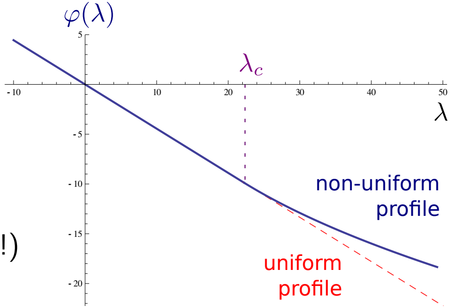

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see also: for LDF of Q [Bodineau, Derrida, PRE **78** 021122] phase transition in WASEP for large dev. (**non-stationary** profile) [Jona-Lasinio *et al.*] generic criterion for instability

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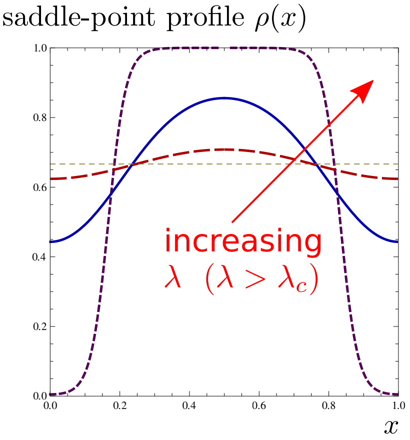
Optimal profiles at $\lambda > \lambda_c$:



Finite-time result on a large time window $[0, t]$

Saddle point evaluation at large L

$$e^{\Psi(s,t)} = \langle e^{-sK} \rangle = \int \mathcal{D}\rho\mathcal{D}\hat{\rho}\; e^{-L\,\mathcal{S}_s[\hat{\rho},\rho]} \sim e^{\overbrace{-L\,\mathcal{S}_s[\hat{\rho},\rho] + \log\det}^{\text{both } \propto t \text{ at } t\to\infty}}$$

The determinant

$q \in \frac{2\pi}{L}\mathbb{Z}$, $\omega \in \frac{2\pi}{t}\mathbb{N}$

$$\log\det = \sum_q \sum_\omega \log \frac{\omega^2 + \Omega_q^2}{\omega^2 + \mathring{\Omega}_q^2}, \qquad \begin{cases} \Omega_q = |q|\sqrt{q^2 - 2\lambda\rho_0(1-\rho_0)} \\ \mathring{\Omega}_q = \Omega_q|_{s=0} \end{cases}$$

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Infinite time

$\sum_\omega = \frac{t}{2\pi}\int d\omega$

$$\log\det = t\sum_q \int \frac{d\omega}{2\pi} \log \frac{\omega^2 + \Omega_q^2}{\omega^2 + \mathring{\Omega}_q^2} = t\sum_q [\Omega_q - \mathring{\Omega}_q]$$

For $\lambda \to \lambda_c^-$, only the mode $q = q_1 \equiv 2\pi/L$ contributes to the singularity

$$\psi(\lambda \to \lambda_c^-)\big|_{\text{singular}} = -\Omega_{q_1} = -\frac{1}{L}\sqrt{\lambda_c - \lambda}$$

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$$\textstyle\sum_\omega = \frac{t}{2\pi}\int d\omega$$

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Finite time $\qquad \sum_\omega = \frac{t}{2\pi}\int d\omega$

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For $\lambda \to \lambda_c^-$, only the mode $q = q_1 \equiv 2\pi/L$ contributes to the singularity

$$\Psi(\lambda \to \lambda_c^-, t)\big|_{\text{singular}} = -\log \frac{\sinh \frac{t}{2L^2}\sqrt{\lambda_c - \lambda}}{\sinh \frac{t}{2L^2}}$$

Finite-time behavior around the Dynamical phase transition

$$\Psi(\lambda \to \lambda_c^-, t)\Big|_{\text{singular}} = -\log \frac{\sinh \frac{t}{2L^2}\sqrt{\lambda_c - \lambda}}{\sinh \frac{t}{2L^2}}$$

Boundary layer $\sqrt{\lambda_c - \lambda} \sim L^2/t$:

- For $\sqrt{\lambda_c - \lambda} \ll L^2/t$:
  exponentially fast convergence of $\Psi(s,t)/t$ to $\psi(s)$
- For $\sqrt{\lambda_c - \lambda} = \mathcal{O}(L^2/t\ )$:
  modified regime
- For $\sqrt{\lambda_c - \lambda} \gg \mathcal{O}(L^2/t\ )$:
  the validity of the computation breaks down (fluctuations are not small)

A surprise: the "**susceptibility**" $\frac{1}{t}\langle K^2\rangle_c = \frac{1}{t}\partial_\lambda^2\Psi(\lambda,t)$ is more divergent (at $\lambda \to \lambda_c$) at finite t than infinite t.

Summary

Microscopic approach:

- ⋆ operator formalism ⋆ XXZ spin chain ⋆ Bethe Ansatz

Macroscopic approach:

- ⋆ MFT, saddle-point method, dynamical phase transition.
- ⋆ finite-time dynamics affected around the transition.

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Questions:

- ⋆ Finite-size crossover around a quantum phase transition? Between:
  - Luttinger Liquid ($s \to -\infty$)
  - Phase-separated ferromagnet ($s \to +\infty$)
- ⋆ XXZ transition not at $\Delta = 1$ but at $\Delta = 1 + \mathcal{O}(L^{-2})$
- ⋆ Finite time for SSEP $\mapsto$ finite temperature for XXZ
- ⋆ Rounding of the transition at finite t

Thank you for your attention!

Collaborators: Cécile Appert-Rolland, Bernard Derrida, Frédéric van Wijland, Juan P. Garrahan, Marc Cheneau

References:

- ⋆ Cécile Appert-Rolland, Bernard Derrida, Vivien Lecomte, Frédéric van Wijland
  Phys. Rev. E **78** 021122 (2008)
- ⋆ Vivien Lecomte, Juan P. Garrahan, Frédéric van Wijland
  J. Phys. A **45** 175001 (2012)
- ⋆ Vivien Lecomte et al.
  work in progress

Sketch of derivation [VL, Garrahan, van Wijland, JPA **45** 175001]

Saddle-point equations for the profile $\rho(x)$ take the form

$$\big(\partial_x \rho(x)\big)^2 + E_P\big(\rho(x)\big) = 0$$

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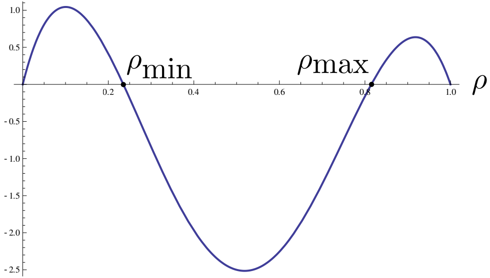
Motion in "time" x of a particle of "position" ρ in a

"Potential energy" $E_P(\rho)$



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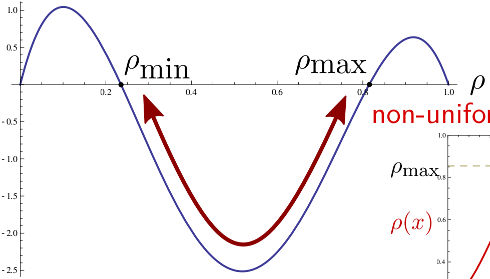
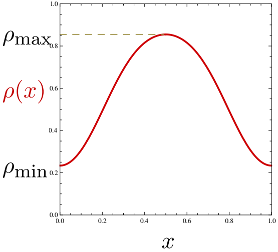


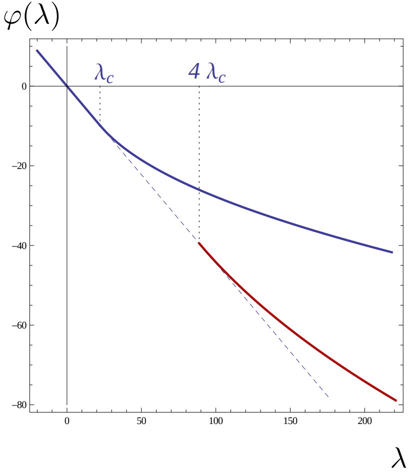
"oscillations" depict the non-uniform profile $\rho(x)$

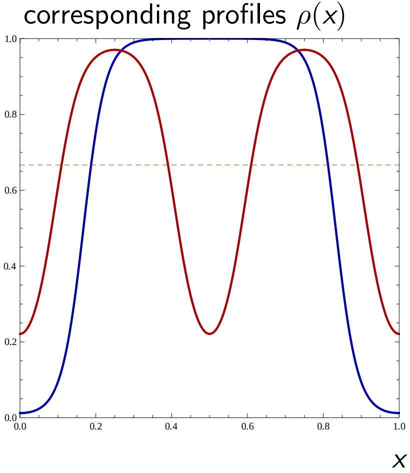


Excitations

[Cheneau, VL, *work in progress*]

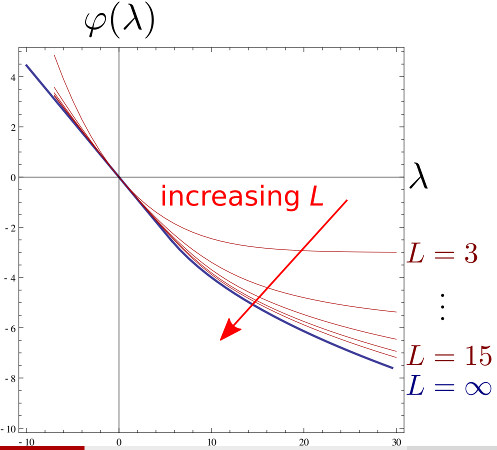
What about solutions with *more than one* kink+anti-kink?





Small sizes: the ground state

Aim: experimental realizations with cold atoms
$\rightarrow$ non-periodic (but isolated, 1D) system
$\rightarrow$ smaller sizes & finite-temperature & excited state



Small sizes: the full spectrum [preliminary!]

$L = 9$ sites
$N_0 = 3$ particles

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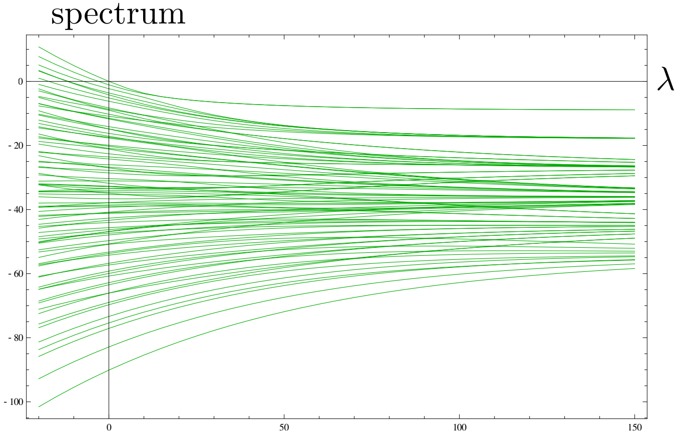

Small sizes: the full spectrum [preliminary!]

$L = 9$ sites
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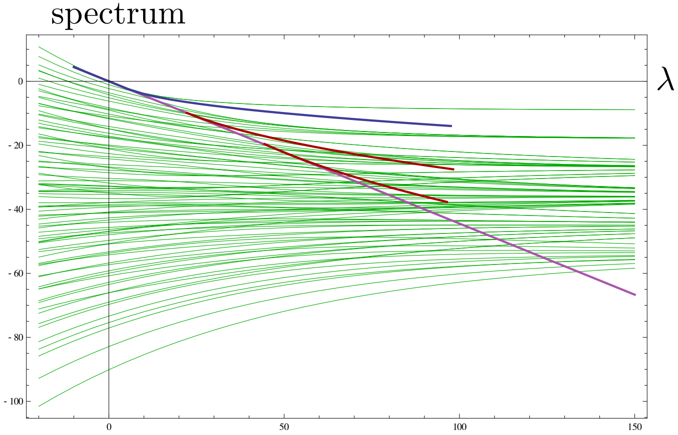
infinite-size ground state
infinite-size excited states



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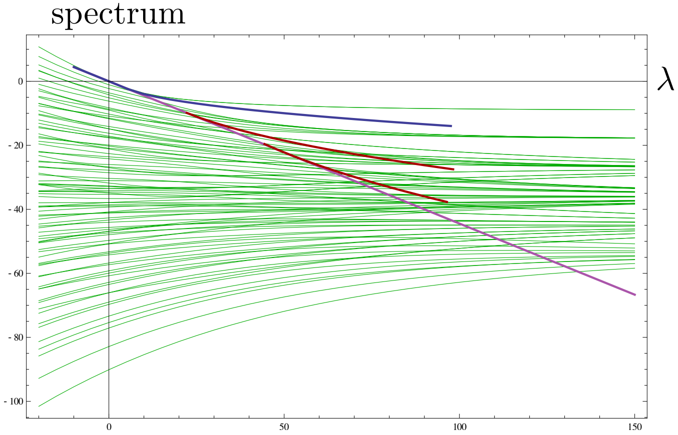

gathering(?) of microscopic eigenvalues $\longrightarrow$ macroscopic ($L = \infty$) states