Large deviations in non-equilibrium 1d systems

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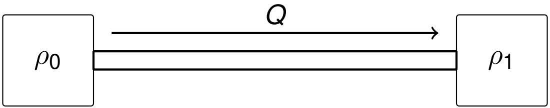
Outline

1. Motivations
2. Microscopic approach
   - Operator approach
   - Bethe Ansatz
3. Macroscopic approach
   - Fluctuating hydrodynamics
   - Finite-size corrections
4. Mapping non-equilibrium to equilibrium
   - For the integrated current
   - For the density profile

Questions



$$\text{Prob}(\mathcal{C}) \propto \exp\left\{-\frac{\text{energy}(\mathcal{C})}{\text{temperature}}\right\}$$ cannot describe

- Non-equilibrium steady-state

$$\text{Prob}[\rho(x)]$$

- Equilibrium fluctuations of dynamical observables

$$\text{Prob}[Q]$$

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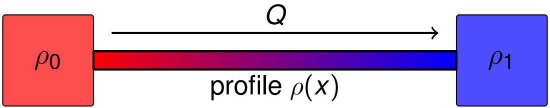

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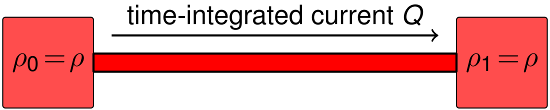

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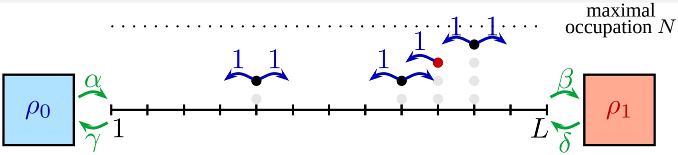
Exclusion Processes



- Configurations: occupation numbers $\{n_i\}$
- Exclusion rule: $0 \leq n_i \leq N$
- Markov evolution

$$\partial_t P(\{n_i\}) = \sum_{n'_i} \left[W(n'_i \to n_i) P(\{n'_i\}) - W(n_i \to n'_i) P(\{n_i\}) \right]$$

- Large deviation function of the time-integrated current Q

$$\langle e^{-sQ} \rangle \sim e^{t\psi(s)} \qquad (\Leftrightarrow \text{determining } P(Q))$$

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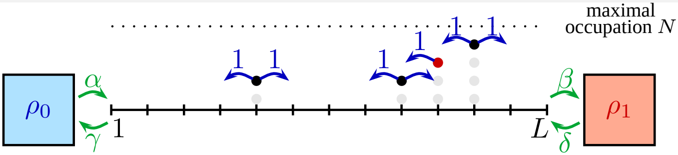

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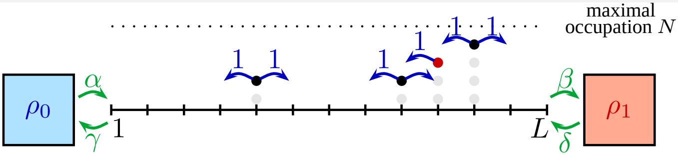

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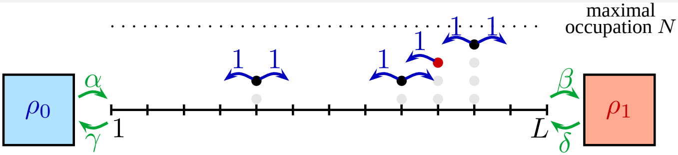
Operator representation [Schütz & Sandow PRE **49** 2726]



$$\partial_t P \;=\; \mathbb{W}\, P$$

$$\mathbb{W} = \sum_{1\leq k\leq L-1} \Big[S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k) \Big]$$
$$+ \alpha \left[S_1^+ - (1-\hat{n}_1) \right] + \gamma \left[S_1^- - \hat{n}_1 \right]$$
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$S^\pm$ and $S^z = \hat{n} - \frac{N}{2}$ are spin operators (with $j = \frac{N}{2}$)

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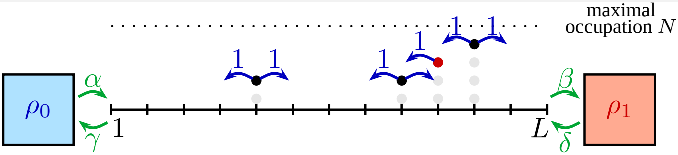

$$\left\langle \mathrm{e}^{-sQ} \right\rangle \sim \mathrm{e}^{t\psi(s)} \qquad \text{with} \qquad \psi(\lambda) = \max \mathrm{Sp}\, \mathbb{W}(s)$$

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Bethe Ansatz method [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

SSEP: maximal occupation $N = 1$
Periodic boundary conditions

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Periodic boundary conditions

Bethe Ansatz:

- eigenvector of components

$$\sum_{\mathcal{P}} \mathcal{A}(\mathcal{P}) \prod_{i=1}^{N} \left[\zeta_{\mathcal{P}(i)}\right]^{x_i}$$

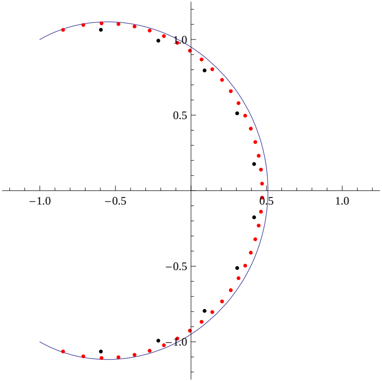
- eigenvalue

$$\psi(s) = -2N + e^{-s}\left[\zeta_1 + \ldots + \zeta_N\right] - e^{s}\left[\frac{1}{\zeta_1} + \ldots + \frac{1}{\zeta_N}\right]$$

- Bethe equations

$$\zeta_i^L = \prod_{\substack{j=1 \\ j\neq i}}^{N} \left[-\frac{1 - 2e^{-s}\zeta_i + e^{-2s}\zeta_i\zeta_j}{1 - 2e^{-s}\zeta_j + e^{-2s}\zeta_i\zeta_j}\right]$$

Bethe Ansatz method [Appert, Derrida, VL, van Wijland, PRE **78** 021122]



Repartition of Bethe roots in the compex plane

Finite-size effects

[Appert, Derrida, VL, van Wijland, PRE **78** 021122]

- large deviation function

$$\psi(s) = \underbrace{L\rho(1-\rho)s^2}_{\text{order 0}} + \underbrace{L^{-2}\mathcal{F}(u)}_{\text{finite-size}} \quad \text{with} \quad u = -\tfrac{1}{2}L^2\rho(1-\rho)s^2$$

- universal function

$$\mathcal{F}(u) = \sum_{k\geq 2} \frac{(-2u)^k B_{2k-2}}{\Gamma(k)\Gamma(k+1)}$$

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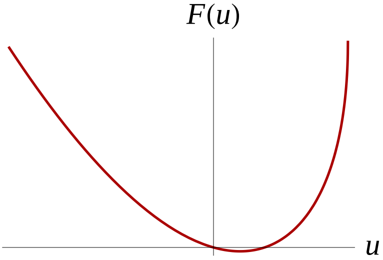
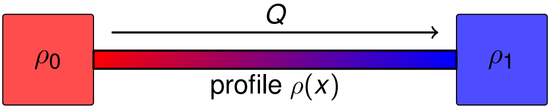


Outline

Macroscopic limit

[Tailleur, Kurchan, VL, JPA **41** 505001]



A reminder: propagator in quantum mechanics

$\langle \text{final} | e^{it\mathbb{H}} | \text{initial} \rangle$

Macroscopic limit

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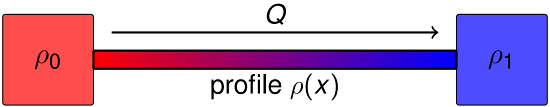

A reminder: propagator in quantum mechanics

$$\langle \text{final}| e^{it\mathbb{H}} |\text{initial}\rangle = \int dz_1 \dots dz_n \langle \text{final}| e^{i\Delta t\mathbb{H}} |\underline{z_n}\rangle \langle \underline{z_{n-1}}| e^{i\Delta t\mathbb{H}} |\underline{z_{n-2}}\rangle \dots$$
$$\dots \langle \underline{z_1}| e^{i\Delta t\mathbb{H}} |\text{initial}\rangle$$

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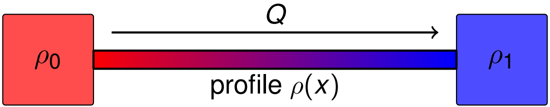

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$$\ldots\langle\underline{z_1}|e^{i\Delta t\mathbb{H}}|\text{initial}\rangle$$

$$= \int \mathcal{D}p\mathcal{D}q \; \exp\{i\tfrac{1}{\hbar}\underbrace{\mathcal{S}[p,q]}_{\text{action}}\}$$

Macroscopic limit

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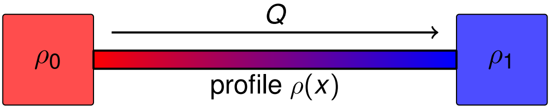

Using $SU(2)$ coherent states:

$$\langle \rho_f | e^{t\mathbb{W}} | \rho_i \rangle = \int_{\rho(0)=\rho_i}^{\rho(t)=\rho_f} \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L \underbrace{\mathcal{S}[\hat{\rho},\rho]}_{\text{action}}\}$$

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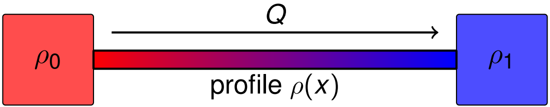

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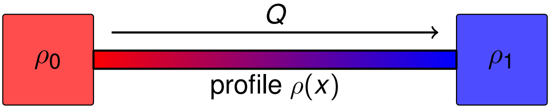

Same $\mathcal{S}_s[\hat{\rho},\rho]$ as the MSR action of the Langevin evolution:

$$\partial_t \rho = -\partial_x \big[-\partial_x \rho + \xi \big]$$
$$\langle \xi(x,t)\xi(x',t')\rangle = \frac{1}{L}\rho(1-\rho)\,\delta(x'-x)\delta(t'-t)$$

Macroscopic limit

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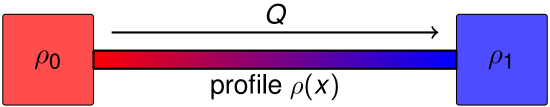

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One recovers the action of fluctuating hydrodynamics

[Spohn, Bertini De Sole Gabrielli Jona-Lasinio Landim]

$\psi(s)$: again [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

Periodic boundary conditions
More general fluctuating hydrodynamics

$$\frac{1}{Lt}\langle Q \rangle \propto D(\rho) \qquad \text{(Fourier's law)}$$

$$\frac{1}{Lt}\langle Q^2 \rangle_c = \sigma(\rho) \qquad \text{(For the SSEP, } \sigma = \rho(1-\rho)\text{)}$$

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Saddle point evaluation

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$$\psi(s) = \underbrace{\tfrac{1}{2}s^2\frac{\langle Q^2\rangle_c}{t}}_{\text{at saddle-point}} + \underbrace{L^{-2}D\mathcal{F}(u)}_{\substack{\text{small fluctuations}\\ \text{(determinant)}}} \qquad \text{with} \quad u = -L^2s^2\frac{\sigma\sigma''}{16D^2}$$

Correspondence between
the (Gaussian) integration of small fluctuations
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More general?

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More general?

Fluctuating hydrodynamics for quantum chains?

With a field [Appert, Derrida, VL, van Wijland, PRE **78** 021122]

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Driving field E

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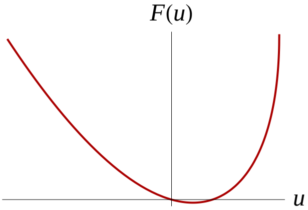

Dynamical phase transition
between
stationnary and non-stationnary
profiles

Outline

Microscopic approach

[Imparato, VL, van Wijland]

Large deviations of the current

$\psi(\lambda) = \max \mathrm{Sp}\, \mathbb{W}(s)$

$$\mathbb{W}(s) = \overbrace{\sum_{1 \le k \le L-1} \vec{S}_k \cdot \vec{S}_{k+1}}^{\text{invariant by rotation}}$$
$$+ \alpha \left[S_1^+ - (1 - \hat{n}_1) \right] + \gamma \left[S_1^- - \hat{n}_1 \right]$$
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Local transformation

$$\mathcal{Q}^{-1}\mathbb{W}(s)\mathcal{Q} = \sum_{1\le k\le L-1} \vec{S}_k \cdot \vec{S}_{k+1} + \alpha' \left[S_1^+ - (1-\hat{n}_1)\right] + \gamma' \left[S_1^- - \hat{n}_1\right] + \underbrace{\delta' \left[S_L^+ e^{s'} - (1-\hat{n}_L)\right] + \beta' \left[S_L^- e^{-s'} - \hat{n}_L\right]}_{\text{describes contact with reservoirs of same densities}}$$

Macroscopic approach [Imparato, VL, van Wijland, arXiv:0904.1478]

$$\langle e^{-sQ}\rangle \sim \int \mathcal{D}\rho\mathcal{D}\hat{\rho}\ \exp\{L\,\mathcal{S}_s[\hat{\rho},\rho]\}$$

Fluctuations ϕ, $\hat{\phi}$ around the saddle

$$\rho(x,t) = \rho_c(x) + \phi(x,t) \qquad\qquad \hat{\rho}(x,t) = \hat{\rho}_c(x) + \hat{\phi}(x,t)$$

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Mapping of non-eq. fluctuations ϕ, $\hat{\phi}$ to eq. fluctuations ϕ', $\hat{\phi}'$

$$\phi(x,t) = (\partial_x\hat{\rho}_c)^{-1}\phi'(x,t) + (\partial_x\rho_c)^{-1}\hat{\phi}'(x,t)$$

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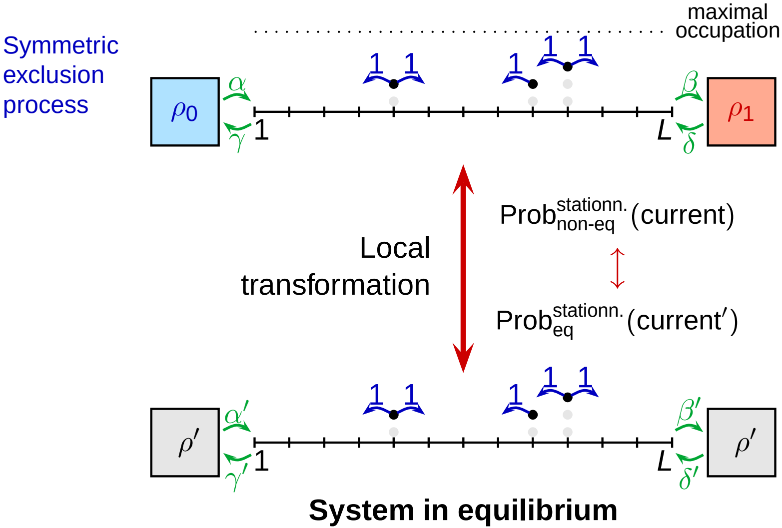
Large deviation function

$$\psi(s)=\underbrace{\frac{1}{L}\mu(sL)}_{\text{saddle}}+\underbrace{\frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(sL)\right)}_{\text{fluctuations}}$$

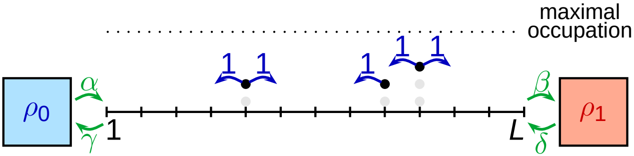
For the current

[Imparato, VL, van Wijland, arXiv:0904.1478]



For the density profile [Tailleur, Kurchan, VL, JPA **41** 505001]

Symmetric exclusion process

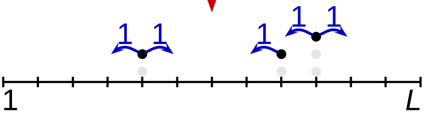


Non-local transformation

$\text{Prob}^{\text{stationn.}}_{\text{non-eq}}(\text{profile})$

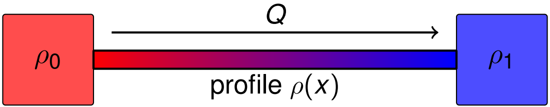
$\updownarrow$

$\text{Prob}^{\text{stationn.}}_{\text{eq}}(\text{profile}')$



System in equilibrium

Non-local mapping [Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

- long-range correlations
- breaking of the time-reversal symmetry

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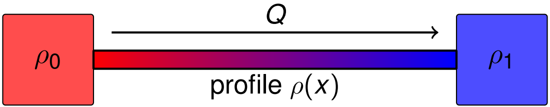
[Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

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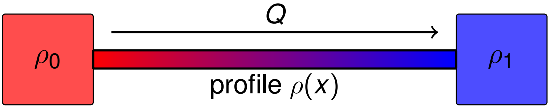
Non-local mapping to equilibrium:

- accounts for the correlations
- $(\text{fixed density})_{\text{eq.}} \longleftrightarrow (\text{density gradient})_{\text{non-eq.}}$
- yields Prob$[\rho(x)]$ through a large deviation function

Non-local mapping [Tailleur, Kurchan, VL, JPA **41** 505001]



Boundary-driven transport model:

- long-range correlations
- breaking of the time-reversal symmetry

Non-local mapping to equilibrium:

- accounts for the correlations
- $(\text{fixed density})_{\text{eq.}} \longleftrightarrow (\text{density gradient})_{\text{non-eq.}}$
- yields Prob[$\rho(x)$] through a large deviation function

$\longrightarrow$ Applies to non-equilibrium quantum chains?

Summary

Macroscopic approach:

- action of fluctuating hydrodynamics
- saddle-point method, instantons
- integration of fluctuations (dynamical phase transition)

Microscopic approach:

- operator formalism
- Bethe Ansatz

→ Eq↔non-eq mapping in higher dimensions?
→ Non-equilibrium 1d quantum transport models?